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Belief function independence: II. The conditional case

Boutheina Ben Yaghlane [a,*], Philippe Smets [b], Khaled Mellouli [a]

[a] *AMID, Université de Tunis, IHEC-Carthage Presidence, Tunis 2016, Tunisia*
[b] *IRIDIA, Université Libre de Bruxelles, Belgium*

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Abstract

In the companion paper [Int. J. Approx. Reasoning 29 (1) (2002) 47], we have emphasized the distinction between non-interactivity and doxastic independence in the context of the transferable belief model. The first corresponds to decomposition of the belief function, whereas the second is defined as irrelevance preserved under Dempster's rule of combination. We had shown that the two concepts are equivalent in the marginal case. We proceed here with the conditional case. We show how the definitions generalize themselves, and that we still have the equivalence between conditional non-interactivity and conditional doxastic independence.

Keywords: Belief functions; Transferable belief model; Conditional non-interactivity; Conditional irrelevance; Conditional doxastic independence

1. Introduction

In many fields of Artificial Intelligence, the notion of conditional independence is considered as very important because it permits to simplify several computational reasoning tasks. Indeed, instead of having to explore a complete

* Corresponding author. Tel.: +216-1-775-948; fax: +216-1-775-944.
E-mail addresses: boutheina.yaghlane@ihec.rnu.tn (B. Ben Yaghlane), psmets@ulb.ac.be (P. Smets), khaled.mellouli@ihec.rnu.tn (K. Mellouli).

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knowledge base of a given complex problem, we organize the problem into simpler components in such a way that we only manipulate the pieces of information having relevance to the question we are interested in.

The concept of probabilistic conditional independence was initially developed by Dawid [6]. More recently and in order to enhance the application of Probability Theory to Artificial Intelligence, Pearl and Paz [19] suggested the connections between conditional independence and graphical representations and proved that the essence of conditional independence can be identified with a common structure consisting of some basic properties of the conditional independence relation, called '*graphoid axioms*'. These axioms convey the simple idea that when we learn an irrelevant fact, the relevance relationships of all other propositions remain unchanged [18].

Several graphoid axioms are also satisfied by embedded multi-valued dependency models in relational databases [12], by conditional independence in Spohn's theory of ordinal conditional functions [14,27], by qualitative conditional independence in Dempster–Shafer theory of belief functions partitions [21], by possibilistic conditional independence [10,13,29], by conditional independence and irrelevance in connection to the theory of closed convex sets of probability measures [5], and by conditional independence in valuation-based systems (VBS) representing many different uncertainty calculi [22].

Unfortunately, these axioms have not received a complete treatment in the literature when related to the theory of belief functions. For this purpose, we study the notion of independence between sets of variables when uncertainty is expressed by belief functions as defined in the context of the transferable belief model (TBM) [25,26]. This study is done in two parts: marginal independence and conditional independence.

In the first part [4], we have discussed different concepts of marginal independence for belief functions and we have clarified the relationships between the concepts of non-interactivity, irrelevance and doxastic independence. Non-interactivity is defined by the 'mathematical' property useful for computation considerations and it means that the joint belief function can be reconstructed from its marginals. Irrelevance is defined by a 'common sense' property based on conditioning, and it means that conditioning the joint belief function on one variable and marginalizing it on the other variable produces a belief function that is the same whatever the conditioning event. Doxastic independence is defined by a particular form of irrelevance, the one preserved under Dempster's rule of combination. The main results of that study is that we have proved that irrelevance alone does not imply non-interactivity. We have also proved that doxastic independence is equivalent to non-interactivity, thus equating the 'common sense' definition with the 'mathematical' one.

In this second part, we extend these concepts of marginal independence for belief functions to conditional case. We particularly discuss the new properties and we show that we still have the equivalence between conditional non-interactivity and conditional doxastic independence. Finally, we present the axiomatic characterization of conditional independence definition for belief functions.

The rest of this paper is organized as follows. In Section 2, we first introduce the necessary notations and terminologies. In Section 3, we recall the definition of probabilistic conditional independence. Then, after extending the definition of evidential and cognitive independence to the conditional case (Section 4), we present our definitions of conditional non-interactivity (Section 5), conditional irrelevance (Section 6) and conditional doxastic independence (Section 7) for belief functions. In Section 8, we discuss the axiomatic characterization of conditional independence definition for belief functions. Finally, in Section 9, we summarize the results achieved in this paper and point out some future directions.

2. Notations and terminologies

The theory of belief functions [20], also known as Dempster–Shafer theory and theory of evidence, aims to model someone's degree of belief. Since this theory was developed, many interpretations have been proposed. Among them, we can distinguish: the lower probability model [15,30], Dempster's model [11], the hint model [16] and the transferable belief model TBM [26]. Like in our companion paper [4], we are only concerned, in this paper, with the TBM.

Most needed definitions and properties have been given in the first part of this paper [4] and the reader is referred to it to find the conventions and the background material on belief functions and the transferable belief model. In this section, we just reproduce the important ones in order to help the reader.

2.1. Sets

When authors discuss about conditional independence, they begin with a set of variables $\mathbf{U} = \{X_1, X_2, X_3, \ldots\}$. Let Ω be a frame of discernment which is the Cartesian product of the variables in $\mathbf{U}$. The concepts of non-interactivity, irrelevance and independence are then defined between only three variables, denoted X, Y, Z, with the understanding that each one represents a variable which domain is the product space of its related variables and the three sets of variables are disjoint.

2.1.1. Notations for sets

We give here some essential set notations.

- By convention, indexed variables like x_i, y_j, z_k denote elements of their domain whereas x, y, z, w denote subsets of their domain.
- If X, Y, Z are three variables, XY denotes $X \times Y$ and XYZ denotes $X \times Y \times Z$.

- For $x \subseteq X, y \subseteq Y$, (x, y) denotes the subset w of XY such that $w = \{(x_i, y_j) : x_i \in x, y_j \in y\}$.
- For $x \subseteq X, y \subseteq Y, z \subseteq Z$, (x, y, z) denotes the subset w of XYZ such that $w = \{(x_i, y_j, z_k) : x_i \in x, y_j \in y, z_k \in z\}$.
- For $x \subseteq X$, $x^{\uparrow XY}$ is the cylindrical extension of x on XY: $x^{\uparrow XY} = (x, Y)$.
- For $w \subseteq \Omega$, $w^{\downarrow X}$ is the projection of w on X: $w^{\downarrow X} = \{x_i : x_i \in X, x_i^{\uparrow\Omega} \cap w \neq \emptyset\}$. This transformation is also called a marginalization.
- For any $w \subseteq XYZ$, we have $w = \bigcup_{z_i \in Z}(A_i, z_i)$, where $A_i \subseteq XY$. Note that A_i may be empty for some i. Equivalently, we have $w = \bigcup_{z_i \in w^{\downarrow Z}}(A_i, z_i)$ in which case $A_i \neq \emptyset$ for all i.
- We use the letters A and B, with or without index, to denote subsets of XZ and YZ, respectively. Let $A \subseteq XZ$ and $B \subseteq YZ$, with

$$A = \bigcup_{z_i \in A^{\downarrow Z}} (x_{i,A}, z_i) \quad \text{and} \quad B = \bigcup_{z_i \in B^{\downarrow Z}} (y_{i,B}, z_i),$$

where the index A (B) in $x_{i,A}$ ($y_{i,B}$) indicates its dependency on A (B). Then

$$A^{\uparrow XYZ} \cap B^{\uparrow XYZ} = \bigcup_{z_i \in A^{\downarrow Z} \cap B^{\downarrow Z}} (x_{i,A}, y_{i,B}, z_i).$$

- We assume that the variables X, Y, Z are 'independent', or 'logically independent', by what we mean that:

$$(x_i, Y, Z) \cap (X, y_j, Z) \cap (X, Y, z_k) \neq \emptyset \quad \forall x_i \in X, y_j \in Y, z_k \in Z.$$

All these definitions are extended to the case where indices are permuted.

2.1.2. Properties of the intersections

In order to avoid any confusion on the domain of the sets, we often indicate it in the first superscript of each set. So A^{XZ} means that $A \subseteq XZ$. The arrows then indicate the extensions ($\uparrow$) and marginalizations ($\downarrow$) to which they are submitted.

Lemma 2.1. $A^{XZ\uparrow XYZ} \cap B^{YZ\uparrow XYZ} \neq \emptyset$ *iff* $A^{XZ\downarrow Z} \cap B^{YZ\downarrow Z} \neq \emptyset$.

Proof. Let $Z_A = A^{XZ\downarrow Z}$ and $Z_B = B^{YZ\downarrow Z}$. We have $A^{XZ} = \bigcup_{z_i \in Z_A}(x_{i,A}, z_i)$ and $B^{YZ} = \bigcup_{z_i \in Z_B}(y_{i,B}, z_i)$, where $x_{i,A} \neq \emptyset$ and $y_{i,B} \neq \emptyset$. Then

$$A^{XZ\uparrow XYZ} \cap B^{YZ\uparrow XYZ} = \begin{cases} \bigcup_{z_i \in Z_A \cap Z_B}(x_{i,A}, y_{i,B}, z_i) \neq \emptyset & \text{if } Z_A \cap Z_B \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases} \qquad \square$$

Lemma 2.2. $(A^{XZ} \cap z^{Z\uparrow XZ})^{\downarrow Z} \neq \emptyset$ *iff* $A^{XZ\downarrow Z} \cap z^Z \neq \emptyset$.

Proof. Direct from Lemma 2.1 when $|Y| = 1$ and B is replaced by z and noting that marginalization does not affect the emptiness status. $\square$

Lemma 2.3. $A^{XZ\uparrow XYZ} \cap B^{YZ\uparrow XYZ} \cap z^{Z\uparrow XYZ} \neq \emptyset$ *iff* $A^{XZ\downarrow Z} \cap B^{YZ\downarrow Z} \cap z^{Z} \neq \emptyset$.

Proof. We have:

$$\begin{aligned}
&A^{XZ\uparrow XYZ} \cap B^{YZ\uparrow XYZ} \cap z^{Z\uparrow XYZ} \neq \emptyset \\
&\quad \text{iff}\ \ B^{YZ\uparrow XYZ} \cap (A^{XZ} \cap z^{Z\uparrow XZ})^{\uparrow XYZ} \neq \emptyset \\
&\quad \text{iff}\ \ B^{YZ\downarrow Z} \cap (A^{XZ} \cap z^{Z\uparrow XZ})^{\downarrow Z} \neq \emptyset \quad \text{by Lemma 2.1} \\
&\quad \text{iff}\ \ B^{YZ\downarrow Z} \cap (A^{XZ\downarrow Z} \cap z^{Z}) \neq \emptyset \quad \text{by Lemma 2.2.} \qquad \square
\end{aligned}$$

2.1.3. The Z-layered rectangles

When studying non-interactivity in a two dimension space, we have introduced the notion of a 'rectangle' as follows.

Definition 2.1 (*Rectangles*). A rectangle in XY is a subset of XY that admits a representation as (x, y) for $x \subseteq X, y \subseteq Y$.

This notion was useful as in case of non-interactivity under m, all focal elements of m are rectangles. This notion can be generalized into a concept of 'Z-layered rectangles'. A Z-layered rectangle (ZLR) is a subset w of XYZ such that for every $z_i \in Z$, its intersection with $z_i^{\uparrow XYZ}$ is a rectangle in XY. Formally, a definition of ZLR is as follows.

Definition 2.2 (*Z-layered rectangles*). A set $w \subseteq XYZ$ is called a ZLR if, for every $z_i \in Z$, $(w \cap z_i^{\uparrow XYZ})^{\downarrow XY}$ is a rectangle in XY.

Lemma 2.4. *A set* $w \subseteq XYZ$ *is a ZLR iff it admits the representation* $w = \bigcup_{z_i \in Z}(x_{i,w}, y_{i,w}, z_i)$, *where* $x_{i,w} \subseteq X, y_{i,w} \subseteq Y$.

Proof. By Definition 2.2, $(w \cap z_i^{\uparrow XYZ})^{\downarrow XY}$ is a rectangle in XY, thus it can be represented as $(x_{i,w}, y_{i,w})$, hence the lemma. $\square$

Fig. 1 presents an example of ZLR. We define ZLR as the set of ZLRs.

Definition 2.3. $ZLR = \{w : w \subseteq XYZ, \forall z_i \in Z, \exists\, x_{i,w} \subseteq X,\ \exists\, y_{i,w} \subseteq Y,$ such that $w = \bigcup_{z_i \in Z}(x_{i,w}, y_{i,w}, z_i)\}$.

In the marginal case, we have shown that the focal elements of m are rectangles in XY when X and Y are non-interactive under m [4, Theorem 3].

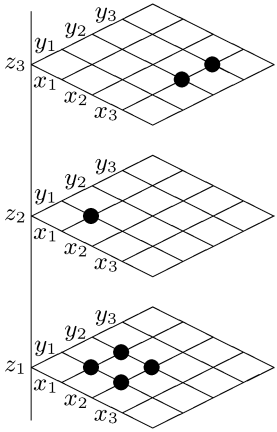

Fig. 1. Example of a *ZLR*, dots indicate its elements.

For the conditional case, we show below that the focal elements of m will be *ZLR*s when X and Y are non-interactive given Z under m.

2.2. Belief functions

2.2.1. Notations

- We use the notation $m^\Omega[x]$ to represent the bba (shorthand for basic belief assignment) m defined on the domain Ω given the belief holder knows (accepts) that x is true (i.e., x holds). The term m can be replaced by bel, pl, q in order to denote the *belief function*, the *plausibility function* and the *commonality function*. The values taken by these functions at $w \subseteq \Omega$ are denoted by $m^\Omega[x](w)$, $\mathrm{bel}^\Omega[x](w)$, $\mathrm{pl}^\Omega[x](w)$, $q^\Omega[x](w)$, respectively. $m^\Omega[x](w)$ is called a *basic belief mass* (bbm).
- When $\omega^* \in \Omega$, $m^\Omega[\omega^*]$ is the result of conditioning m^Ω on ω^* by Dempster's rule of conditioning (without the normalization). So, among others:

$$m^\Omega[\omega^*](\omega) = \begin{cases} \sum_{v \subseteq \overline{\omega^*}} m^\Omega(\omega^* \cup v) & \text{if } \omega \subseteq \omega^*, \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{pl}^\Omega[\omega^*](\omega) = \mathrm{pl}^\Omega(\omega^* \cap \omega) \quad \forall \omega \subseteq \Omega.$$

As we do not normalize, the problem of conditioning on an event with 0 plausibility is not an issue. If $\mathrm{pl}^\Omega(\omega^*) = 0$, one gets $m^\Omega[\omega^*](\emptyset) = 1$.

- In the TBM, none of these functions is necessarily normalized. When we want to get the normalized forms, we use the upper-cases notations M, Bel, Pl, Q. These normalized functions are obtained by dividing the unnormalized functions by the factor $1 - m(\emptyset)$ (putting $M(\emptyset) = 0$, $\mathrm{Bel}(\emptyset) = 0$ and $Q(\emptyset) = 1$).

- The $\oplus$ symbol represents Dempster's rule of combination in its normalized form and $\circledcap$ represents the conjunctive combination, i.e., the same operation as Dempster's rule of combination except the normalization (the division by $1 - m(\emptyset)$) is not performed. The conjunctive combination rule can be written equivalently as:

$$m_{1\circledcap 2}(w) = m_1 \circledcap m_2(w) = \sum_{w_1, w_2 \subseteq \Omega, w_1 \cap w_2 = w} m_1(w_1) m_2(w_2).$$

The next formula is very useful:

$$f_{1\circledcap 2}(w) = \sum_{w^* \subseteq \Omega} f_1[w^*](w) m_2(w^*) \quad \forall w \subseteq \Omega, \tag{1}$$

where $f \in \{m, \mathrm{bel}, \mathrm{pl}, q\}$ and $f_1[w^*]$ is the result of the unnormalized conditioning of f_1 on $w^* \subseteq \Omega$ (see [23]).
- $\mathrm{pl}_1 \circledcap \mathrm{pl}_2$ represents the plausibility function obtained from $m_1 \circledcap m_2$, where m_1 and m_2 are the bba's related to pl_1 and pl_2, respectively (and similarly with bel and q).
- The set of belief functions defined on Ω is denoted by BF_Ω.
- By abuse of language, we may omit the Ω index and we will write statements like $m \in \mathrm{BF}_\Omega$ to mean that the belief function associated with m belongs to BF_Ω.
- When convenient, bba's m on Ω are represented by the list of pairs (w, x) where w is a focal element of m (a subset of Ω with a non-null bbm), and $x = m(w)$. So $((w_1, .4), (w_2, .6))$ represents the bba m^Ω on Ω with $m^\Omega(w_1) = .4$ and $m^\Omega(w_2) = .6$, and $w_1 \subseteq \Omega$ and $w_2 \subseteq \Omega$.

2.2.2. Marginalization and combination

We present some of the relations described by Shenoy concerning marginalization and used in this paper. Their labels are those of [22].

Lemma 2.5 (M1). *Order of deletion*

$$m^{XYZ\downarrow YZ\downarrow Z} = m^{XYZ\downarrow XZ\downarrow Z}.$$

Lemma 2.6 (CM1). *Combination and marginalization*

$$(m^{UV} \circledcap m^{VW})^{\downarrow UV} = m^{UV} \circledcap m^{VW\downarrow V}.$$

Lemma 2.7. *Given* M1, $m^{XYZ\downarrow YZ\downarrow Z} = m^{XYZ\downarrow Z}$.

Proof. See [22, p. 210]. $\square$

3. Probabilistic conditional independence

First, we present the meaning of the conditional independence concept in the probability theory. Suppose three variables X, Y and Z, and let the space $\Omega = XYZ$. Let P be a distribution on XYZ. We write $X \perp\!\!\!\perp_P Y|Z$ to denote that, X and Y are *conditionally independent* given Z, with respect to P.

The usual definition of $X \perp\!\!\!\perp_P Y|Z$ is in terms of the factorization of the conditional joint probability distribution on XY given Z. We say that X and Y are *conditionally independent* given Z, with respect to P^{XYZ}, if and only if $\forall x \subseteq X,\ \forall y \subseteq Y,\ \forall z_i \in Z$,

$$P^{XYZ}[z_i]^{\downarrow XY}(x,y) = P^{XYZ}[z_i]^{\downarrow X}(x)P^{XYZ}[z_i]^{\downarrow Y}(y), \tag{2}$$

where $P^{XYZ}[z_i]^{\downarrow XY}$ is the conditional probabilities of P^{XYZ} on XY, given z_i and $P^{XYZ}[z_i]^{\downarrow X}$ and $P^{XYZ}[z_i]^{\downarrow Y}$ are, respectively, the conditional probabilities of P^{XYZ} on X and Y, given z_i.

There is another equivalent definition, which is more intuitive, that is: X and Y are *conditionally independent* given Z, with respect to P^{XYZ}, if and only if $\forall x \subseteq X$, $\forall y \subseteq Y$, $z_i \in Z$,

$$P^{XYZ}[y,z_i]^{\downarrow X}(x) = P^{XYZ}[z_i]^{\downarrow X}(x), \tag{3}$$

where $P^{XYZ}[y,z_i]^{\downarrow X}$ is the conditional probability of P^{XYZ} on X given y and z_i.

Like the marginal probabilistic case, this second definition can be interpreted as *conditional irrelevance* and it means that once the value of Z is specified, any further information about Y is irrelevant to the uncertainty about X.

As far as the two definitions turn out to be equivalent, the distinction between the factorization and the irrelevance approach is not essential, and often it is not even considered.

4. Evidential and cognitive independence

In the marginal case [4], we have presented the notions of evidential independence and cognitive independence for belief functions. These notions have been first introduced by Shafer for the marginal case. In addition, it is shown in [20] that evidential independence implies cognitive independence, but not the reverse. In this section, we only consider evidential independence.

In the multivariate framework, Kong [17] studied the conditional case. These definitions are based on normalized belief functions, the concept of unnormalized belief functions was not yet introduced. He defined the notion of evidential conditional independence of belief functions as follows (remember that the variables X, Y and Z are always pairwise disjoint subsets of variables (see Section 2)):

Definition 4.1 (*Evidential conditional independence*). Let X, Y and Z be three variables. X and Y are conditionally independent given Z with respect to Bel^{XYZ} if and only if:

$$\mathrm{Bel}^{XYZ}[z_i]^{\downarrow XY} = \mathrm{Bel}^{XYZ}[z_i]^{\downarrow X} \oplus \mathrm{Bel}^{XYZ}[z_i]^{\downarrow Y}. \tag{4}$$

When Z is not specified this becomes marginal evidential independence of X and Y.

Almond [1, p. 114] defines a *strong* conditional independence as:

Definition 4.2 (*Strong conditional independence*). Let X, Y and Z be three variables. X and Y are (strongly) conditionally independent given Z with respect to Bel^{XYZ} if and only if

$$\mathrm{Bel}^{XYZ} = \mathrm{Bel}^{XYZ\downarrow XZ} \oplus \mathrm{Bel}^{XYZ\downarrow YZ}. \tag{5}$$

It happens that these two definitions are not equivalent? The next example presents a bba m which satisfy Kong's definition (Eq. (4)) but not Almond's definition (Eq. (5)).

Example 4.1 (*Showing the inequality of Kong's and Almond's definitions*). Let $X = \{0, 1\}$, $Y = \{0, 1\}$ and $Z = \{0, 1\}$ be three binary spaces. Table 1 presents the data for an example on XYZ where the three variables are binary. Elements of XYZ are denoted as abc where the three bits a, b, c refer to the value of X, Y and Z, respectively. The dots like in 00. means that both the elements 000 and 001 belong to the set. So ... is XYZ. The three sets of columns present each the list of focal sets and their masses. The first pair presents the initial bba m defined on XYZ. The second pair presents its marginalization on XZ followed by an extension back on XYZ. The third does the same on YZ. When recombining these two bba (the marginal on XZ and on YZ) by the conjunctive combination rule, the mass given to 00. is $0.45 \times 0.50 = 0.225$ which is different from $m(00.) = 0.20$. Hence X and Y do not satisfy Almond's definition (see Table 1).

Table 1
Data for bba's on XYZ with focal elements and their masses

XYZ	m	XYZ	$m^{\downarrow XZ\uparrow XYZ}$	XYZ	$m^{\downarrow YZ\uparrow XYZ}$
00.	0.20	0..	0.45	.0.	0.50
00. ∪ .00	0.05	0.. ∪ ..0	0.05	...	0.50
00. ∪ .01	0.05	0.. ∪ ..1	0.05		
.0.	0.20	...	0.45		
0..	0.25				
...	0.25				

Table 2
Data of Table 1 where initial bba m is conditioned on $Z=0$ and $Z=1$, respectively, and trivially projected on XY space

Given $Z=0$		Given $Z=1$	
XY	$m[Z=0]^{\downarrow XY}$	XY	$m[Z=1]^{\downarrow XY}$
00	0.25	00	0.25
.0	0.25	.0	0.25
0.	0.25	0.	0.25
..	0.25	..	0.25

Nevertheless if we condition on $Z=0$ or on $Z=1$, we find that the bba's on XY, presented in Table 1, satisfy the independence property required by Kong's definition. So m satisfy the Kong's definition (see Table 2).

Therefore the two definitions are not identical. In fact the second is more demanding than the first that looks for independence after conditioning only on the elements of Z.

Note that the Kong's definition and Almond's definition are based on normalized belief functions. When we tolerate unnormalized belief functions, the term $\mathrm{bel}^{XYZ\downarrow Z}$ must be added and the definition becomes as follows:

Definition 4.3 (*Strong conditional independence*). Let X, Y and Z be three variables. X and Y are (strongly) conditionally independent given Z with respect to bel^{XYZ} if and only if

$$\mathrm{bel}^{XYZ} \circledcirc \mathrm{bel}^{XYZ\downarrow Z} = \mathrm{bel}^{XYZ\downarrow XZ} \circledcirc \mathrm{bel}^{XYZ\downarrow YZ}. \tag{6}$$

This definition turns out to be equivalent to what we call hereafter conditional non-interactivity. In the following sections, we present our definitions of conditional non-interactivity, conditional irrelevance and conditional doxastic independence.

5. Conditional non-interactivity

Preamble. The proofs of the theorems are given in Appendix A. In addition, many proofs are highly simplified when a matricial notation is used. This notation being unusual, we do not use it in the core of the paper, but we relegate it in Appendix B. The proofs using such notations are therefore also put in Appendix B.

5.1. Definition of conditional non-interactivity

We focus now on the decompositional independence definition for belief functions. This definition is represented by the non-interactivity that is a mathematical property useful for calculus considerations. For the full study of the marginal non-interactivity concept for belief functions, the reader can be referred to [3,4].

However, for the definition of the conditional non-interactivity for belief functions (see also [2]), we start from the belief on the joint product XYZ. We marginalize it on XZ and also on YZ. We combine these two marginal belief functions and we want it to be equal to the initial one (on XYZ) combined with its marginal on Z.

This last term results from the fact that the marginals on XZ and on YZ both contain the marginal on Z and this last marginal is thus double counted when combining the marginals on XZ and on YZ. This term corresponds to the $\mathrm{pl}^{XY}(X,Y)$ term encountered when defining marginal independence (see [4, relation (6)]). The formal definition is given as follows:

Definition 5.1 (*Conditional non-interactivity*). Given three variables X, Y and Z, and $m^{XYZ} \in \mathrm{BF}_{XYZ}$, X and Y are conditionally non-interactive given Z with respect to m^{XYZ}, denoted by $X \perp_{m^{XYZ}} Y|Z$, if and only if

$$m^{XYZ} \bigcirc\!\!\!\!\cap\, m^{XYZ\downarrow Z} = m^{XYZ\downarrow XZ} \bigcirc\!\!\!\!\cap\, m^{XYZ\downarrow YZ}. \tag{7}$$

This definition of conditional non-interactivity (7) corresponds to Shenoy' factorization (see [22], Lemma 3.1 (5) p. 215). It can also be reformulated in terms of commonality functions as shown by Studeny in [28].

Theorem 5.1. *$X \perp_{m^{XYZ}} Y|Z$ iff for all $w \subseteq XYZ$,*

$$q^{XYZ}(w)\; q^{XYZ\downarrow Z}(w^{\downarrow Z}) = q^{XYZ\downarrow XZ}(w^{\downarrow XZ})\; q^{XYZ\downarrow YZ}(w^{\downarrow YZ}). \tag{8}$$

Proof. See Appendix A. □

5.2. Links with marginal non-interactivity

The marginal case corresponds to the conditional case when $|Z| = 1$. Then Definition 5.1 becomes equal to the one used in the marginal case [4, Definition 4]. When $|Z| = 1$, we have:

- $XYZ = XY,\ XZ = X,\ YZ = Y$,
- $w^{\downarrow Z} = Z$ if $w \neq \emptyset$ and $= \emptyset$ otherwise.
- When $w \neq \emptyset$, $q^{XYZ\downarrow Z}(w^{\downarrow Z})$ becomes the sum of all bba's m^{XY} given to the non-empty subsets of XY, which is equal to $\mathrm{pl}^{XY}(XY)$.

Eq. (8) becomes then:

$$q^{XY}(w)\ \mathrm{pl}^{XY}(XY) = q^{XY\downarrow X}(w^{\downarrow X})\ q^{XY\downarrow Y}(w^{\downarrow Y})$$

for all $w \subseteq XY, w \neq \emptyset$. The case where $w = \emptyset$ is trivially satisfied as $q(\emptyset) = 1$ for any commonality function. This relation is the definition of marginal non-interactivity. So our definition degrades nicely into the marginal case when $|Z| = 1$, as it should.

5.3. *Conditional non-interactivity and marginals*

It is interesting to note that Studeny has an objection about the definition of conditional non-interactivity [1] in the framework of Dempster–Shafer theory [28]. Indeed, he notices that the definition based on Eq. (7) is *not consistent with marginalization*. It may happen that for two bba's $m_1 \in \mathrm{BF}_{XZ}$ and $m_2 \in \mathrm{BF}_{YZ}$ that share the same marginal on Z (i.e., $m_1^{\downarrow Z} = m_2^{\downarrow Z}$) there exists no bba m^{XYZ} on XYZ such that $m^{XYZ\downarrow XZ} = m_1$, $m^{XYZ\downarrow YZ} = m_2$ and $X \perp_{m^{XYZ}} Y|Z$. The next example illustrates this objection.

Example 5.1 (*From Studeny, personal communication*). Consider $X = Y = Z = \{u, v\}$ and define the bba's over XZ and YZ as follows:

$$m^{XZ}(\{(u,v),(v,v)\}) = 0.5,$$
$$m^{XZ}(\{(u,v),(v,u)\}) = 0.5,$$
$$m^{YZ}(\{(u,v),(v,v)\}) = 0.5,$$
$$m^{YZ}(\{(u,v),(v,u)\}) = 0.5.$$

There is no bba m over XYZ such that m^{XZ} and m^{YZ} are its marginals and X and Y are conditionally non-interactive given Z with respect to m.

Let $m^Z = m^{XZ\downarrow Z} = m^{YZ\downarrow Z}$. Its focal elements and associated bbm are: $((v, .5), (Z, .5))$. In order to get non-interactivity, the bba m must satisfy

$$q^{XYZ}(w) = \frac{q^{XYZ\downarrow XZ}(w^{\downarrow XZ})q^{XYZ\downarrow YZ}(w^{\downarrow YZ})}{q^{XYZ\downarrow Z}(w^{\downarrow Z})} \quad \forall w \subseteq XYZ.$$

The resulting 'q' function is not a commonality function as its associated 'bba' is:

$$((X,Y,v),.25),\ ((X,u,v),.25),\ ((u,Y,v),.25),$$
$$(\{(u,u,v),(v,v,u)\},.5),\ ((u,u,v),-.25),$$

[1] Studeny uses the term 'conditional independence' rather than 'conditional non-interactivity'.

which does not correspond to a belief function as one of the 'masses' is negative.

This example illustrates that two bba's m^{XZ} and m^{YZ} that share the same marginal m^{Z} on Z are not the marginal of some bba m^{XYZ} such that X and Y are conditionally non-interactive given Z (i.e. $X \perp_{m^{XYZ}} Y|Z$).

Nevertheless the next theorem shows that, for any m^{XZ} and m^{YZ}, X and Y are non-interactive given Z under $m^{XYZ} = m^{XZ} \circledcirc m^{YZ}$. The only subtlety is that m^{XZ} and m^{YZ} are *not* the marginals of m^{XYZ} on XZ and YZ, respectively. This property provides in fact a convenient way to build belief functions that satisfy non-interactivity. Just take any pair of bba's m^{XZ} and m^{YZ} and combine them conjunctively, the result is a bba under which X and Y are conditionally non-interactive given Z.

Theorem 5.2. *Let m^{XZ} and m^{YZ} be two bba on XZ and YZ, respectively. Let $m = m^{XZ} \circledcirc m^{YZ}$. Then $X \perp_m Y|Z$.*

Proof. See Appendix A. □

Example 5.2 (*continuation of Example 5.1*). The focal elements and related bbm for $m = m^{XZ} \circledcirc m^{YZ}$ are:

$$((X, Y, v), .25),\ ((X, u, v), .25),\ ((u, Y, v), .25), (\{(u, u, v), (v, v, u)\}, .25).$$

Its marginals are:

for $m^{\downarrow XZ}$: $((X, v), .5),\ ((u, v), .25), (\{(u, v), (v, u)\}, .25)$
for $m^{\downarrow YZ}$: $((Y, v), .5),\ ((u, v), .25), (\{(u, v), (v, u)\}, .25)$
and for $m^{\downarrow Z}$: $((v, .75), (Z, .25))$.

These bba's satisfy relation (8), thus we have $X \perp_m Y|Z$.

5.4. Conditional non-interactivity and Z-layered rectangles

When we have treated the marginal non-interactivity between two random variables X and Y, we have proved that when X and Y are non-interactive, with respect to m^{XY}, then the focal elements of m^{XY} belong to Rect_{XY} [4, Theorem 3]. We proceed now with the same idea applied to the conditional case and we show that the focal elements of $m = m^{XYZ}$ belong to the set of ZLRs (see Section 2.1.3).

Theorem 5.3. *If $X \perp_m Y|Z$, then the focal elements of m belong to the set of ZLRs.*

Proof. See Appendix A. □

5.5. Conditions for non-interactivity

The simple fact that the focal elements belong to the set of *ZLR*s is not sufficient to imply conditional non-interactivity. The next example illustrates such a case.

Example 5.3 (*ZLR without non-interactivity*). Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z\}$, and $m^{XYZ}(x_1, y_1, z) = .5$, $m^{XYZ}(X, Y, z) = .5$. By construction, this bba belongs to *ZLR*. To be non-interactive, m^{XYZ} must satisfy (7), and in particular $m^{XYZ}(x_1, y_1, z) = m^{XYZ\downarrow XZ}(x_1, z)m^{XYZ\downarrow YZ}(y_1, z) = .5 \times .5 = .25$. So $m^{XYZ} \in ZLR$ without satisfying non-interactivity.

In order to get non-interactivity, we must add some proportionality constraints, like those presented in the next theorem. It must be emphasized that in the definition of non-interactivity (see Theorem 5.1), relation (8) among the commonality functions had to be true for all w in XYZ, whereas here relation (9) is required only on the w in the set of *ZLR*s. Theorem 5.4 is very useful as we will later show that doxastic conditional independence is equivalent to the properties of this theorem, and thus doxastic conditional independence will be proved to be equivalent to conditional non-interactivity.

Theorem 5.4. *Suppose a bba* m^{XYZ}. *The next assertions are equivalent*
1. $X \perp_{m^{XYZ}} Y|Z$.
2. *The focal elements of* m^{XYZ} *belong to ZLR and*

$$\forall w \in ZLR, \quad q^{XYZ}(w)\ q^{XYZ\downarrow Z}(w^{\downarrow Z}) = q^{XYZ\downarrow XZ}(w^{\downarrow XZ})\ q^{XYZ\downarrow YZ}(w^{\downarrow YZ}). \tag{9}$$

Proof. See Appendix A. □

6. Conditional irrelevance

Before presenting the definition of conditional irrelevance for belief functions, we explain the idea of two belief functions on XYZ that share the same marginals on Z after having been conjunctively combined with a given bba m defined on XYZ.

The underlying idea is a problem of belief state distinguishability. Suppose two agents who hold beliefs on XYZ. Suppose you can only observe the beliefs held by these two agents on Z (thus the marginal on Z of their bba's). If these two marginal bba's are equal, You cannot distinguish between the beliefs held by the two agents, even though their beliefs on XYZ may be different. One way to distinguish the two beliefs is to present to the two agents a new piece of evidence which induces the bba m on XYZ. This last m is then combined conjunctively with the initial bba's. The marginalizations on Z can still be equal, or not, this depending on m. So one way to distinguish between belief states which can only be observed on Z is by producing various m, and comparing the marginals on Z of the combination.

For a given m on XYZ, we can consider all the belief functions on XYZ which are indistinguishable on Z. These bba's describe belief states that cannot be distinguished after having been conjunctively combined with m by only observing their marginals on Z. Thus m creates an equivalence class on the set of belief functions defined on XYZ.

6.1. Indistinguishability on Z under m

Let $R^Z(m)$ denotes the set of *pairs* of belief functions on XYZ that are indistinguishable on Z under m. Its formal definition is as follows:

Definition 6.1 (*Indistinguishability on Z under m*). For any bba $m, m_1, m_2 \in \mathrm{BF}_{XYZ}$, $(m_1, m_2) \in R^Z(m)$ iff $(m \circledcirc m_1)^{\downarrow Z} = (m \circledcirc m_2)^{\downarrow Z}$.

In particular, we will use this concept of indistinguishability when $m \in \mathrm{BF}_{XYZ}$ and $m_1, m_2 \in \mathrm{BF}_{YZ}$ what is just a particular case of the definition. The reason will be that we will define conditional irrelevance as the fact that the belief on XZ is influenced by the belief on YZ only through the impact of this last belief on Z, and not on the details on how it is distributed on YZ. In the following example, we illustrate the concept of indistinguishability.

Example 6.1 (*Indistinguishability*). Let $X = \{0, 1\},\ Y = \{0, 1\},\ Z = \{0, 1\}$. The elements of $\Omega = XYZ$ are denoted by $\omega_1, \ldots, \omega_8$. Table 3 presents the elements of XYZ and their X, Y, Z values. For instance, element ω_3 has values $X = 0$, $Y = 1, Z = 0$.

Let us consider the set YZ with four singletons, and thus any bba on YZ can have at most 16 focal sets. In Table 4, the top row indicates the bbm x_i for $i = 1, \ldots, 16$ that can be given to the 16 subsets of YZ, with $x_i \geqslant 0$ and their sum is 1. The index i refers to the subsets of YZ according to the convention described in the next four rows of Table 4. The four singletons of YZ are given

Table 3
The elements of XYZ and their X, Y, Z values

Z	0	0	0	0	1	1	1	1
Y	0	0	1	1	0	0	1	1
X	0	1	0	1	0	1	0	1
Ω	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8

Table 4
The 16 bbm's of a bba defined on YZ and their focal sets

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}
$\{\omega_1, \omega_2\}$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
$\{\omega_3, \omega_4\}$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
$\{\omega_5, \omega_6\}$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
$\{\omega_7, \omega_8\}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

in the left side column. In the column x_i, the 1's indicate the singletons that belong to the subset which mass is x_i. So x_7 is the mass given to the focal element $\{\omega_3, \omega_4, \omega_5, \omega_6\}$ of XYZ by the bba m defined on YZ and extended on XYZ.

Let m^{XYZ} be a bba with $m^{XYZ}(\omega_1, \omega_2, \omega_3, \omega_4) = m^{XYZ}(\omega_1, \omega_5) = 1/2$. The set of bba's m on YZ that are pairwise indistinguishable on Z under m^{XYZ} is built in Table 5.

The second line, in each part of Table 5, indicates the elements of XYZ to which the mass $m^{XYZ}(\omega_1, \omega_5) = 1/2$ is transferred by the conjunctive combination of m^{XYZ} and m. So $0.5 \cdot x_3$ is transferred to ω_5. The last two lines, to be read together, indicate the indices of the elements of XYZ to which the mass $m^{XYZ}(\omega_1, \omega_2, \omega_3, \omega_4) = 1/2$ is transferred by the conjunctive combination of m^{XYZ} and m. So $0.5 \cdot x_{13}$ is transferred to $\{\omega_1, \omega_2, \omega_3, \omega_4\}$.

After marginalization of $m^{XYZ} \circledcirc m$ on XZ, the bbm of m^Z is given by:

$$m^Z(\emptyset) = (x_1 + x_2 + x_5 + x_6 + x_1 + x_2 + x_3 + x_4)/2, \tag{10}$$

$$m^Z(\omega_1, \omega_2, \omega_3, \omega_4) = (x_9 + x_{10} + x_{13} + x_{14} + x_5 + x_6 + \ldots + x_{16})/2, \tag{11}$$

$$m^Z(\omega_5, \omega_6, \omega_7, \omega_8) = (x_3 + x_4 + x_7 + x_8)/2, \tag{12}$$

$$m^Z(\omega_1, \omega_2, \ldots, \omega_8) = (x_{11} + x_{12} + x_{15} + x_{16})/2. \tag{13}$$

Any set of non-negative values for the x_i's that add to 1 and such that the four sums (10)–(13) are constant will generate the same bba on Z after being combined with m^{XYZ}. Hence they are indistinguishable on Z under m^{XYZ}. The next two bba's are such examples.

- $x_7 = m_1(\omega_3, \omega_4, \omega_5, \omega_6) = 1/2,\ x_{13} = m_1(\omega_1, \omega_2, \omega_3, \omega_4) = 1/2$. The focal elements of $(m^{XYZ} \circledcirc m_1)^{\downarrow Z}$ are $\{\omega_5\}, \{\omega_1\}, \{\omega_3, \omega_4\}\{\omega_1, \omega_2, \omega_3, \omega_4\}$, each with a mass 1/4. The projection of these focal elements on Z are $\{\omega_5, \omega_6, \omega_7, \omega_8\}$ with a mass 1/4 and $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ with a mass 3/4.
- $x_8 = m_2(\omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8) = 1/2,\ x_{10} = m_2(\omega_1, \omega_2, \omega_7, \omega_8) = 1/2.$ The focal elements of $(m^{XYZ} \circledcirc m_2)^{\downarrow Z}$ are $\{\omega_5\}, \{\omega_1\}, \{\omega_3, \omega_4\}\{\omega_1, \omega_2\}$, each with a mass 1/4. The projection of these focal elements on Z are $\{\omega_5, \omega_6, \omega_7, \omega_8\}$ with a mass 1/4 and $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ with a mass 3/4.

So the bba's $(m^{XYZ} \circledcirc m_1)^{\downarrow Z}$ and $(m^{XYZ} \circledcirc m_2)^{\downarrow Z}$ are equal, hence m_1 and m_2 are indistinguishable on Z under $m^{XYZ} : (m_1, m_2) \in R^Z(m^{XYZ})$.

Table 5
Finding the set of bba's indistinguishable under m^{XYZ}

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}
$\{\omega_1, \omega_5\}$	$\emptyset$	$\emptyset$	ω_5	ω_5	$\emptyset$	$\emptyset$	ω_5	ω_5	ω_1	ω_1	ω_1, ω_5	ω_1, ω_5	ω_1	ω_1	ω_1, ω_5	ω_1, ω_5
$\{\omega_1, \omega_2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	ω_3	ω_3	ω_3	ω_3	ω_1	ω_1	ω_1	ω_1	ω_1, ω_2	ω_1, ω_2	ω_1, ω_2	ω_1, ω_2
$\{\omega_3, \omega_4\}$					ω_4	ω_4	ω_4	ω_4	ω_2	ω_2	ω_2	ω_2	ω_3, ω_4	ω_3, ω_4	ω_3, ω_4	ω_3, ω_4

We show now what indistinguishability implies about the involved bba's. In the following theorem, m_1 and m_2 belong to BF_{YZ} and not BF_{XYZ}. As this is all we need, more general properties are useless.

Theorem 6.1. *Let $m \in \mathrm{BF}_{XYZ}$, and $m_1, m_2 \in \mathrm{BF}_{YZ}$. The following assertions are equivalent.*

1. $(m_1, m_2) \in R^Z(m)$.
2. $$\sum_{C\subseteq YZ} \mathrm{pl}(z^{\uparrow XYZ} \cap C^{\uparrow XYZ}) m_1(C) = \sum_{C\subseteq YZ} \mathrm{pl}(z^{\uparrow XYZ} \cap C^{\uparrow XYZ}) m_2(C) \quad \forall z \subseteq Z. \tag{14}$$

Proof. See Appendix A. □

6.2. Definition of conditional irrelevance

Let $m \in \mathrm{BF}_{XYZ}$. Suppose that we study the impact of any bba $m_i \in \mathrm{BF}_{YZ}$ on our belief on XZ, i.e., we study $(m \circledcirc m_i)^{\downarrow XZ}$. Suppose the impact of m_i on m is fully captured by its impact on Z. By that we mean that the impact of m_i defined on YZ and the impact of any other m_j defined on YZ with $(m_i, m_j) \in R^Z(m)$ are equal when it comes to the belief induced on XZ. Equivalently it means that all that counts for what regards our beliefs on XZ after we combine m with m_i is the belief induced by $m \circledcirc m_i$ on Z. Further details on the beliefs on YZ are irrelevant.

In that case, we say that Y is *conditionally irrelevant* to X given Z with respect to m. Formally, we have the following definition:

Definition 6.2 (*Conditional irrelevance*). Let $m \in \mathrm{BF}_{XYZ}$. Y is conditionally irrelevant to X given Z with respect to m, denoted by $\mathrm{IR}_m(X, Y|Z)$, if and only if for all $m_1, m_2 \in \mathrm{BF}_{YZ}$ with $(m_1, m_2) \in R^Z(m)$ we have

$$(m \circledcirc m_1)^{\downarrow XZ} = (m \circledcirc m_2)^{\downarrow XZ}. \tag{15}$$

Note that our definition of conditional irrelevance is symmetrical as proven here:

Theorem 6.2. $\mathrm{IR}_m(X, Y|Z)$ *iff* $\mathrm{IR}_m(Y, X|Z)$.

Proof. See Appendix B. □

Example 6.2 (*Continuation of* Example 6.1). The bba m^{XYZ} introduced in Example 6.1 does not satisfy conditional irrelevance. Indeed there are two bba's m_1 and m_2 on YZ which pair belongs to $R^Z(m^{XYZ})$ but with:

Table 6
The focal elements and their corresponding masses

bba	Mass	Focal
m_1	$x_3 = 1/2$	$\{\omega_5, \omega_6\}$
m_1	$x_9 = 1/2$	$\{\omega_1, \omega_2\}$
m_2	$x_7 = 1/2$	$\{\omega_3, \omega_4, \omega_5, \omega_6\}$
m_2	$x_5 = 1/2$	$\{\omega_3, \omega_4\}$

$$(m^{XYZ} \circledcirc m_1^{\uparrow XYZ})^{\downarrow XZ} \neq (m^{XYZ} \circledcirc m_2^{\uparrow XYZ})^{\downarrow XZ}.$$

The two bba's are:

- $m_1(\omega_5, \omega_6) = m_1(\omega_1, \omega_2) = 1/2$, or equivalently $x_3 = x_9 = 1/2$.
- $m_2(\omega_3, \omega_4, \omega_5, \omega_6) = m_2(\omega_3, \omega_4) = 1/2$, or equivalently $x_7 = x_5 = 1/2$.

This is an example where indistinguishability does not imply conditional irrelevance. In Table 6, the two m_1 lines present the two focal sets of m_1 and their masses. The focal sets that result from their combination with m^{XYZ} (each mass is 1/4) are presented in Table 7. We continue with the marginalization of these four focal sets on Z, and on XZ. The next two lines concerns m_2. The computation of the masses is given in Table 7. It is to be noticed that the two bba's obtained after projection on Z are equal (indistinguishability), whereas they are not on XZ (no conditional irrelevance).

It shows that even though $(m_1, m_2) \in R^Z(m^{XYZ})$, their marginalization on XZ after combination with m^{XYZ} are not equal, and thus we do not have $\mathrm{IR}_{m^{XYZ}}(X, Y|Z)$.

6.3. Links with marginal irrelevance

The definition of marginal irrelevance is given by relation (10) in Definition 5 of the companion paper [4]: for $m \in \mathrm{BF}_{XY}$, $\mathrm{IR}_m(X, Y)$ iff $\mathrm{pl}[y]^{\downarrow X} \propto \mathrm{pl}^{\downarrow X}$.

We show that our definition of conditional irrelevance reduces into that definition when $|Z| = 1$.

Theorem 6.3 (The marginal case). *If* $|Z| = 1$, $\mathrm{IR}_m(X, Y|Z) = \mathrm{IR}_m(X, Y)$.

Proof. See Appendix B. □

So the definition degrades nicely. One may wonder why we need the concept of indistinguishability and why simpler requirements based on plausibility functions would not be sufficient as when $|Z| = 1$. They are just not sufficient, and this results from the interactions one can build on a product space and that are not apparent when one space degenerates.

Table 7
Indistinguishability and conditional irrelevance

Focal	m^{XYZ}		$m_i \circledcap m^{XYZ}$		$m_i \circledcap m^{XYZ}$	
	$\{\omega_1, \omega_5\}$	$\{\omega_1, \omega_2, \omega_3, \omega_4\}$	$\downarrow Z$		$\downarrow XZ$	
$\{\omega_5, \omega_6\}$	ω_5	$\emptyset$	$\omega_5, \omega_6, \omega_7, \omega_8$	$\emptyset$	ω_5, ω_7	$\emptyset$
$\{\omega_1, \omega_2\}$	ω_1	ω_1, ω_2	$\omega_1, \omega_2, \omega_3, \omega_4$	$\omega_1, \omega_2, \omega_3, \omega_4$	ω_1, ω_3	$\omega_1, \omega_2, \omega_3, \omega_4$
$\{\omega_3, \omega_4, \omega_5, \omega_6\}$	ω_5	ω_3, ω_4	$\omega_5, \omega_6, \omega_7, \omega_8$	$\omega_1, \omega_2, \omega_3, \omega_4$	ω_5, ω_7	$\omega_1, \omega_2, \omega_3, \omega_4$
$\{\omega_3, \omega_4\}$	$\emptyset$	ω_3, ω_4	$\emptyset$	$\omega_1, \omega_2, \omega_3, \omega_4$	$\emptyset$	$\omega_1, \omega_2, \omega_3, \omega_4$

6.4. Conditional irrelevance and non-interactivity

In the following example, we show that conditional irrelevance does not imply conditional non-interactivity between variables. We use the same example as in the marginal case [4], but as far as the conditional case use a more general concept of indistinguishability, we feel necessary to reproduce the example within this larger framework.

Example 6.3. Let XYZ be as defined by Table 3. Table 8 presents a very symmetrical bba m^{XYZ} on XYZ, where all focal elements are subsets of $\{\omega_1, \omega_2, \omega_3, \omega_4\}$. m^{XYZ} satisfies the irrelevance constraints but not the non-interactivity ones.

In this table, for each subset ω of XYZ listed in column 1, column 2 presents the value of m^{XYZ}. To show that m^{XYZ} satisfies conditional irrelevance, we build Table 9. As far as $\mathrm{bel}^{XYZ}(\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 1$, the masses of m^{YZ} on YZ relevant for their combination with m^{XYZ} are those obtained by conditioning the bba's $m^{YZ} \in \mathrm{BF}_{YZ}$ on $\{\omega_1, \omega_2, \omega_3, \omega_4\}$. Then only four masses must be

Table 8
Irrelevance and non-interactivity

For ω such that:	$m^{XYZ}(\omega)$
$\omega = \emptyset$ or $\omega \not\subseteq \{\omega_1, \omega_2, \omega_3, \omega_4\}$	.00
$\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\}$	.15
$\omega \in \{(\omega_1, \omega_2), (\omega_3, \omega_4), (\omega_1, \omega_3), (\omega_2, \omega_4)\}$	.00
$\omega \in \{(\omega_1, \omega_4), (\omega_2, \omega_3)\}$	.04
$\omega \subseteq \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\lvert\omega\rvert = 3$	.02
$\omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$	.24

Table 9
m^{XYZ} satisfies conditional irrelevance

m^{XYZ}		m^{YZ}				$(m^{XYZ} \circledcap m^{YZ})^{\downarrow XZ}$		
	Focal Set	x_1 $\emptyset$	x_2 34	x_3 12	x_4 1234	x_2 12	x_3 34	x_4 1234
.15	1	$\emptyset$	$\emptyset$	1	1	$\emptyset$	13	13
.15	2	$\emptyset$	$\emptyset$	2	2	$\emptyset$	24	24
.15	3	$\emptyset$	3	$\emptyset$	3	13	$\emptyset$	13
.15	4	$\emptyset$	4	$\emptyset$	4	24	$\emptyset$	24
.04	14	$\emptyset$	4	1	14	24	13	1234
.04	23	$\emptyset$	3	2	23	13	24	1234
.02	123	$\emptyset$	3	12	123	13	1234	1234
.02	124	$\emptyset$	4	12	124	24	1234	1234
.02	134	$\emptyset$	34	1	134	1234	13	1234
.02	234	$\emptyset$	34	2	234	1234	24	1234
.24	1234	$\emptyset$	34	12	1234	1234	1234	1234

considered, denoted x_1, x_2, x_3, x_4, with for instance $x_2 = m^{YZ}[\{\omega_1, \omega_2, \omega_3, \omega_4\}] \times (\{\omega_3, \omega_4\})$.

Columns 1 and 2 present the masses and focal elements of m^{XYZ}. The focal sets are represented by the indices of the elements of XYZ (see Table 3) which belong to them. So 14 means $\{\omega_1, \omega_4\}$. Columns 3–6 consider the four masses of m^{YZ} and present the focal elements where they will be transferred after conjunctive combination with m^{XYZ}. The last rightmost columns indicate the subsets where the masses of the combination are transferred after marginalization on XZ.

For what concerns the marginalization on Z, all masses given to the non-empty set are projected on $\{\omega_1, \omega_2, \omega_3, \omega_4\}$. Hence to get indistinguishability on Z under m_{XYZ}, their sum must be fixed. So indistinguishability is satisfied if the masses x_1, x_2, x_3, x_4 are such that:

$$0.7(x_2 + x_3) + x_4 = c \tag{16}$$

is constant. Any pair of bba on YZ that satisfy the constraint for a given c belongs to $R^Z(m^{XYZ})$.

Given c, we consider now what are the marginalization on XZ of their combination with m^{XYZ}, what can be evaluated with the three rightmost columns of Table 9.

On $\{\omega_1, \omega_3\}$, the mass is: $(.15 + .04 + .02)x_2 + (.15 + .04 + .02)x_3 + (.15 + .15)x_4 = .21(x_2 + x_3) + .30x_4 = .3c$

On $\{\omega_2, \omega_4\}$, the mass is: $.21(x_2 + x_3) + .30x_4 = .3c$.

On $\{\omega_1, \omega_2, \omega_3, \omega_4\}$, the mass is: $.28(x_2 + x_3) + .40x_4 = .4c$.

Thus, for every $m \in \mathrm{BF}_{YZ}$ such that $(m^{XYZ} \circledcirc m^{YZ\uparrow XYZ})^{\downarrow Z}$ is equal, (i.e., that satisfies Eq. (16)), $(m^{XYZ} \circledcirc m^{YZ\uparrow XYZ})^{\downarrow XZ}$ is equal. Therefore conditional irrelevance is satisfied.

Nevertheless, non-interactivity is not satisfied as some focal elements of m^{XYZ} are not ZLRs (Theorem 5.3), e.g. $m^{XYZ}(\omega_1, \omega_4) > 0$.

6.5. Conditioning on Z and conditional irrelevance

Just as in the marginal case, irrelevance is not equal to non-interactivity. We will need an extra constraint on irrelevance, the *irrelevance preservation under combination* IRP$\circledcirc$, in order to get the equality between these two concepts.

When we prove that non-interactivity is equal to doxastic independence, i.e., irrelevance plus irrelevance preservation under combination, we will use a particular belief function, the bba characterizing a conditioning on $z_0 \subseteq Z$, that satisfies the irrelevance conditions.

Theorem 6.4. *If* $m(z_0^{\uparrow XYZ}) = 1$ *for* $z_0 \subseteq Z$, *then* $\mathrm{IR}_m(X, Y|Z)$.

Proof. See Appendix B. □

7. Conditional doxastic independence

In the probabilistic framework, it can be easily proved that independence and irrelevance concepts are equivalent. However, in the belief functions framework, the situation is not as simple, irrelevance alone does not imply independence. In the marginal case [4], we have defined that two variables are *doxastically independent* when they are irrelevant and this irrelevance is preserved under Dempster's rule of combination. Then we prove that non-interactivity and doxastic independence are equivalent.

In this section, we show that the notion of doxastic independence [2] defined in the marginal case can be extended to the conditional case. We discuss also the relationship between conditional doxastic independence and ZLRs. Finally, we state two theorems establishing the equivalence between conditional doxastic independence and conditional non-interactivity.

7.1. Irrelevance preservation under conjunctive combination

Just as in the marginal case, we feel that conditional doxastic independence requires not only the conditional irrelevance property, but that property should be preserved when combining two belief functions that satisfy it. The idea fits with the next scenario: if two agents claim that X and Y are conditionally doxastically independent given Z, then this conditional independence should be preserved when the belief functions representing the agents' beliefs are conjunctively combined.

So conditional doxastic independence is irrelevance plus irrelevance preservation under conjunctive combination, denoted IRP$\bigcirc\!\!\!\!\cap$. Formally, the last property is defined as follows:

Definition 7.1 (*Irrelevance preservation under conjunctive combination*). Given $m_1, m_2 \in \mathrm{BF}_{XYZ}$, we say they satisfy IRP$\bigcirc\!\!\!\!\cap$ if $\mathrm{IR}_{m_1}(X, Y|Z)$ and $\mathrm{IR}_{m_2}(X, Y|Z)$, imply $\mathrm{IR}_{m_1 \bigcirc\!\!\!\!\cap\, m_2}(X, Y|Z)$.

7.2. Definition of conditional doxastic independence

The notion of doxastic independence defined in the marginal case can be extended to the conditional case by the following definition.

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[2] We use here the term 'doxastic independence' for making the distinction between probabilistic independence and belief function independence. In Greek, 'doxein' means 'to believe'.

Definition 7.2 (*Conditional doxastic independence*). Given three variables X, Y, Z, and $m \in \mathrm{BF}_{XYZ}$. The variables X and Y are *doxastically independent* given Z with respect to m, denoted by $X \perp\!\!\!\perp_m Y|Z$, if and only if m satisfies

- $\mathrm{IR}_m(X, Y|Z)$,
- $\forall m_0 \in \mathrm{BF}_{XYZ} : \mathrm{IR}_{m_0}(X, Y|Z) \Rightarrow \mathrm{IR}_{m \circledcap m_0}(X, Y|Z)$.

7.3. Conditional doxastic independence and Z-layered rectangles

In the marginal case ($|Z| = 1$), we have defined $X \perp\!\!\!\perp_m Y$ with $m \in \mathrm{BF}_{XY}$ as irrelevance and irrelevance preservation under conjunctive combination (i.e., $X \perp\!\!\!\perp_m Y$ iff $\mathrm{IR}_m(X, Y)$ and $\mathrm{IRP}\circledcap$). We have shown that $X \perp\!\!\!\perp_m Y$ implies that the focal elements of m are rectangles on XY. We proceed here by showing that we have the same thing with conditional case.

Theorem 7.1. *If $X \perp\!\!\!\perp_m Y|Z$, then the focal elements of m belong to ZLR.*

Proof. See Appendix A. □

When $|Z| = 1$, the *ZLR*s reduce to rectangles on XY.

7.4. Conditional doxastic independence and conditional non-interactivity

The equivalence between conditional doxastic independence and conditional non-interactivity is given by the next theorems.

Theorem 7.2. *$X \perp\!\!\!\perp_m Y|Z$ implies $X \perp_m Y|Z$.*

Proof. See Appendix A. □

Theorem 7.3. *$X \perp_m Y|Z$ implies $X \perp\!\!\!\perp_m Y|Z$.*

Proof. See Appendix A. □

8. Axiomatic characterization

As we said before, reasoning systems should take into account conditional independence considerations in order to get an efficient performance. Conditional independence is given by the *conditional independence relations* [6,18], which successfully depict our intuition about how dependencies should update in response to new pieces of information.

In this section, we first introduce the conditional independence relations in an abstract way (Section 8.1). We emphasize, essentially, the intuitive meaning of these relations. After this, we present the characterization of conditional independence definition for belief functions (Section 8.2).

8.1. Conditional independence relations

The concept of conditional independence has been well studied in probability theory (see, for instance, [6,7,18], etc.). This study of probabilistic conditional independence has resulted in the identification of several properties that should be satisfied by any relationship which attempts to capture the intuitive notion of independence.

Recently, several researchers propose to treat conditional independence (CI) without any connection to probability theory. For this purpose, CI is presented as an abstract concept [7]. This approach leads to a better understanding of the CI properties, and then facilitates efficient computations in reasoning systems. In this context, the intuitive meaning of the (abstract) conditional independence is given in terms of irrelevance. Suppose three (sets of) variables X, Y and Z. When we say that X is *conditionally independent* to Y given Z (written $X \perp\!\!\!\perp Y|Z$), we mean that once the value of Z has been specified, any further information about Y is irrelevant to the uncertainty about X. In order to capture the main properties of this abstract notion, some axioms are proposed [18,7]:

- A1: *Symmetry*: $X \perp\!\!\!\perp Y|Z \Rightarrow Y \perp\!\!\!\perp X|Z$.
- A2: *Decomposition*: $X \perp\!\!\!\perp (Y,W)|Z \Rightarrow X \perp\!\!\!\perp Y|Z$.
- A3: *Weak union*: $X \perp\!\!\!\perp (Y,W)|Z \Rightarrow X \perp\!\!\!\perp Y|(Z,W)$.
- A4: *Contraction*: $X \perp\!\!\!\perp Y|Z$ and $X \perp\!\!\!\perp W|(Z,Y) \Rightarrow X \perp\!\!\!\perp (Y,W)|Z$.
- A5: *Intersection*: $X \perp\!\!\!\perp Y|(Z,W)$ and $X \perp\!\!\!\perp W|(Z,Y) \Rightarrow X \perp\!\!\!\perp (Y,W)|Z$.

A dependency model is called *semi-graphoid* if it verify Axioms A1–A4, and *graphoid* if it satisfy Axioms A1–A5. It is well known that probabilistic independence relation is a *semi-graphoid*, and it is a *graphoid* if the probability is strictly positive (in this case A5 is satisfied).

8.2. Belief function conditional independence relations

The properties of conditional independence can be considered as a set of rules useful to infer new independence relations from an initial set. They are also important when we need a *graphical representation* of dependencies.

When studying the concept of conditional independence in valuation-based systems (VBS), [3] Shenoy have proved that the conditional independence

[3] VBS is an axiomatic framework capable of representing many different uncertainty theories.

concept satisfies the graphoid axioms [22]. As the theory of belief functions is one particular uncertainty theory of the VBS framework, we proceed similarly and we provide in Theorem 8.1 the conditional independence properties for belief functions that also satisfy graphoid axioms. For this purpose, we use the definition of conditional non-interactivity.

Theorem 8.1. *Let X, Y, Z and W be disjoint subsets of a set of variables U, and a mass m over the product space. The following properties are satisfied*:

Symmetry $X \perp_m Y|Z \iff Y \perp_m X|Z.$

Decomposition $X \perp_m Y \cup W|Z \Rightarrow X \perp_m Y|Z.$

Weak Union $X \perp_m Y \cup W|Z \Rightarrow X \perp_m Y|W \cup Z.$

Contraction $X \perp_m Y|Z$ and $X \perp_m W|Y \cup Z \Rightarrow X \perp_m Y \cup W|Z.$

Intersection $X \perp_m Y|Z \cup W$ and $X \perp_m W|Z \cup Y \Rightarrow X \perp_m Y \cup W|Z.$

Proof. The corresponding proofs showing the validity of these properties for any belief function are given in Appendix A. □

Furthermore, it is easy to prove that Dawid redundancy property is satisfied.

Theorem 8.2. *For any* $m \in \mathrm{BF}_{XY}, X \perp_m Y|X.$

Proof. See Appendix A. □

9. Conclusion

Like for other uncertainty formalisms, the concept of conditional independence is also important in belief functions theory. After the first part of this study [4] in which we have studied the marginal belief function independence, we proceed similarly and we propose, in this paper, an extension of the marginal case to the conditional case. For this purpose, we present the definitions of:

- *Conditional non-interactivity*: the joint belief function can be rebuilt from its marginals.
- *Conditional irrelevance*: the belief on XZ depends on any belief over YZ only through the impact of the last belief function on Z.
- *Conditional irrelevance preservation under conjunctive combination rule*: if two belief functions satisfy conditional irrelevance, then their conjunctive combination satisfies also conditional irrelevance.

- *Conditional doxastic independence*: defined as conditional irrelevance that is preserved under conjunctive combination rule.

The major result is that conditional non-interactivity and conditional doxastic independence are equivalent. Furthermore, we show that belief function conditional non-interactivity satisfy the graphoid axioms.

In future work, we will investigate:

- the existence and the properties of *conditional products* [9] for belief function theory,
- the links between our concept of conditional doxastic independence and the concept of *separoid* recently introduced in [8],
- the impact of conditional doxastic independence with respect to its graphical representation and the propagation of information in this structure.

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Appendix A. Proofs of theorems

Proof of Theorem 5.1. Relation (8) is just a rewriting of relation (7) using the property that $q_1 \circledcirc q_2(A) = q_1(A)q_2(A)$ for all A in the domain of the q's, and

$$q^{XYZ\downarrow XZ}(w^{\downarrow XZ}) = q^{XYZ\downarrow XZ\uparrow XYZ}(w),$$
$$q^{XYZ\downarrow YZ}(w^{\downarrow YZ}) = q^{XYZ\downarrow YZ\uparrow XYZ}(w),$$
$$q^{XYZ\downarrow Z}(w^{\downarrow Z}) = q^{XYZ\downarrow Z\uparrow XYZ}(w). \qquad \square$$

Proof of Theorem 5.2. By CM1 (Lemma 2.6) we have:

$$m^{\downarrow XZ} = (m^{XZ} \circledcirc m^{YZ})^{\downarrow XZ} = m^{XZ} \circledcirc m^{YZ\downarrow Z}, \tag{A.1}$$
$$m^{\downarrow YZ} = (m^{XZ} \circledcirc m^{YZ})^{\downarrow YZ} = m^{XZ\downarrow Z} \circledcirc m^{YZ}. \tag{A.2}$$

By M1 (Lemma 2.5), we have: $m^{\downarrow Z} = (m^{\downarrow XZ})^{\downarrow Z}$ (see [22, p. 210]) and from CM1 (Lemma 2.6), we have:

$$m^{\downarrow Z} = (m^{\downarrow XZ})^{\downarrow Z} = (m^{XZ} \circledcirc m^{YZ\downarrow Z})^{\downarrow Z} = m^{XZ\downarrow Z} \circledcirc m^{YZ\downarrow Z}. \tag{A.3}$$

So $m^{\downarrow Z} = m^{XZ\downarrow Z} \circledcirc m^{YZ\downarrow Z}$. Thus using relations (A.1)–(A.3), we get:

$$m^{\downarrow XZ} \circledcirc m^{\downarrow YZ} = m^{XZ} \circledcirc m^{YZ\downarrow Z} \circledcirc m^{XZ\downarrow Z} \circledcirc m^{YZ} = m^{XZ} \circledcirc m^{YZ} \circledcirc m^{\downarrow Z} = m \circledcirc m^{\downarrow Z}$$

hence $X \perp_m Y|Z$. $\square$

Proof of Theorem 5.3. Let A be a focal element of $m^{\downarrow XZ}$ and $A = \bigcup_{z_i \in A^{\downarrow Z}} (x_{i,A}, z_i)$.

Let B be a focal element of $m^{\downarrow YZ}$ and $B = \bigcup_{z_i \in B^{\downarrow Z}} (y_{i,B}, z_i)$.

Then $A^{\uparrow XYZ} \cap B^{\uparrow XYZ}$ is a focal element of $m^{\downarrow XZ} \circledcirc m^{\downarrow YZ}$ and $A^{\uparrow XYZ} \cap B^{\uparrow XYZ} = \bigcup_{z_i \in A^{\downarrow Z} \cap B^{\downarrow Z}} (x_{i,A}, y_{i,B}, z_i)$ which belongs to ZLR.

So all focal elements of $m^{\downarrow XZ} \circledcirc m^{\downarrow YZ}$ belong to ZLR.

Let C be a focal element of $m^{\downarrow Z}$. Trivially, $C = \bigcup_{z_i \in C} (z_i)$ and $C^{\uparrow XYZ} = \bigcup_{z_i \in C} (X, Y, z_i)$.

Let w be a focal element of m and $w = \bigcup_{z_i \in w^{\downarrow Z}} (C_{i,w}, z_i)$, where $C_{i,w} \subseteq XY$.

The focal elements of $m \circledcirc m^{\downarrow Z}$ are $C^{\uparrow XYZ} \cap w = \bigcup_{z_i \in w^{\downarrow Z} \cap C} (C_{i,w}, z_i)$ and they must belong to ZLR, as $m \circledcirc m^{\downarrow Z}$ and $m^{\downarrow XZ} \circledcirc m^{\downarrow YZ}$ share the same focal elements (the two belief functions are equal).

So there exist (x_i, y_i) with $x_i \subseteq X, y_i \subseteq Y$ such that $C_{i,w} = (x_i, y_i)$, and thus the focal elements of m belong to ZLR. $\square$

Proof of Theorem 5.4. $1 \Rightarrow 2$. Let $X \perp_{m^{XYZ}} Y|Z$. By Theorem 5.3, the focal elements of m^{XYZ} are ZLRs. By Theorem 5.1, q^{XYZ} satisfies relation (9) for all $w \subseteq XYZ$. Hence 1 implies 2.

$2 \Rightarrow 1$. We show that if the focal elements of m^{XYZ} are ZLRs, then relation (9) implies relation (8). Consider relation (9). Let $w \subseteq XYZ$.

Then $q^{XYZ}(w)$ is the sum of all bbm which contain w. Let $r_w \subseteq XYZ$ be the smallest ZLR in XYZ that contains w. Let $w = \bigcup_{z_i \in Z} (w_i, z_i)$, where $w_i \in XY$. Then r_w is uniquely defined as $r_w = \bigcup_{z_i \in Z} (w_i^{\downarrow X}, w_i^{\downarrow Y}, z_i)$. The only focal elements that contain w are those that contain r_w, so $q^{XYZ}(w) = q^{XYZ}(r_w)$.

We also have $q^{XYZ\downarrow XZ}(w^{\downarrow XZ}) = q^{XYZ\downarrow XZ}(r_w^{\downarrow XZ})$ as only the supersets of $r_w^{\downarrow XZ}$ are supersets of $w^{\downarrow XZ}$ with non-zero bbm. And similarly with the other two marginalizations.

The relation (8) holds iff the relation (9) holds. Therefore $2 \Rightarrow 1$. $\square$

Proof of Theorem 6.1. By definition, assertion 1 means $(m_1 \circledcirc m)^{\downarrow Z} = (m_2 \circledcirc m)^{\downarrow Z}$. By Lemmas 2.7 and 2.6, each term, with $i = 1, 2$, can be rewritten as:

$$(m_i \circledcirc m)^{\downarrow Z} = (m_i \circledcirc m)^{\downarrow YZ \downarrow Z} = (m_i \circledcirc m^{\downarrow YZ})^{\downarrow Z}.$$

By Eq. (1), we have for $B \subseteq YZ$,

$$\begin{aligned} \mathrm{pl}_i \circledcirc \mathrm{pl}^{\downarrow YZ}(B) &= \sum_{C \subseteq YZ} \mathrm{pl}^{\downarrow YZ}[C](B) m_i(C) = \sum_{C \subseteq YZ} \mathrm{pl}^{\downarrow YZ}(B \cap C) m_i(C) \\ &= \sum_{C \subseteq YZ} \mathrm{pl}((B \cap C)^{\uparrow XYZ}) m_i(C) = \sum_{C \subseteq YZ} \mathrm{pl}(B^{\uparrow XYZ} \cap C^{\uparrow XYZ}) m_i(C). \end{aligned}$$

We also have:

$$(\mathrm{pl}_i \circledcirc \mathrm{pl}^{\downarrow YZ})^{\downarrow Z}(z) = \mathrm{pl}_i \circledcirc \mathrm{pl}^{\downarrow YZ}(z^{\uparrow YZ}) = \sum_{C \subseteq YZ} \mathrm{pl}(z^{\uparrow XYZ} \cap C^{\uparrow XYZ}) m_i(C).$$

Requiring $(m \circledcirc m_1)^{\downarrow Z} = (m \circledcirc m_2)^{\downarrow Z}$ is equivalent to requiring:

$$(\mathrm{pl}_1 \circledcirc \mathrm{pl}^{\downarrow YZ})^{\downarrow Z}(z) = (\mathrm{pl}_2 \circledcirc \mathrm{pl}^{\downarrow YZ})^{\downarrow Z}(z) \quad \forall z \subseteq Z$$

what is equivalent to:

$$\sum_{C \subseteq YZ} \mathrm{pl}(z^{\uparrow XYZ} \cap C^{\uparrow XYZ}) m_1(C) = \sum_{C \subseteq YZ} \mathrm{pl}(z^{\uparrow XYZ} \cap C^{\uparrow XYZ}) m_2(C) \quad \forall z \subseteq Z.$$

what proves the equivalence between assertions 1 and 2. □

Proof of Theorem 7.1. Let $m \in \mathrm{BF}_{XYZ}$ satisfies $\mathrm{IR}_m(X, Y|Z)$. Let $z_0 \in Z$. We know that the bba $m_0(z_0^{\uparrow XYZ}) = 1$ also satisfies conditional irrelevance (see Theorem 6.4). IRP$\circledcirc$ implies that we still have $X \perp\!\!\!\perp_{m \circledcirc m_0} Y|Z$. But $m \circledcirc m_0$ is the bba defined on $X \times Y \times z_0$ that corresponds to the conditioning of m on z_0 (by Dempster's rule of conditioning). Furthermore the results correspond to the case $|Z| = 1$. So we know that the focal elements of $m[z_0] = m \circledcirc m_0$ are rectangles on YX. It means that m must be so that for all $z_i \in Z$, the focal elements of $m[z_i]$ are rectangles in $X \times Y \times z_i$. Therefore the focal elements of m belong to the set of ZLR.

Suppose it was not the case, and $m(C) > 0$ for $C \subseteq XYZ$. Then there is at least one $z_j \in Z$ such that $C \cap z_j^{\uparrow XYZ}$ is not a rectangle. Then $m[z_j](C \cap z_j^{\uparrow XYZ})$ is then positive (as it contains $m(C)$). But in that case we do not have doxastic independence after conditioning on z_j, contrary of what we had shown. □

Proof of Theorem 7.2. Both requirements of Theorem 5.4 are satisfied: the ZLR requirement is proved in Theorem 7.1 and the commonality requirement is proved in Theorem B.8 in Appendix B. □

Proof of Theorem 7.3. We first prove that $X \perp_m Y|Z$ implies $\mathrm{IR}_m(X, Y|Z)$. For that, we use the removal operator of Shenoy [22], here denoted $\overline{\circledcirc}$. By definition $m_3 = m_1 \overline{\circledcirc} m_2$ is the bba m_3 such that $m_3 \circledcirc m_2 = m_1$. It is just the inverse of $\circledcirc$.

If $X \perp_m Y|Z$ for $m \in BR_{XYZ}$, then by definition we have

$$m = m^{\downarrow XZ} \circledcirc m^{\downarrow YZ} \overline{\circledcirc} m^{\downarrow Z}.$$

Suppose $m_1, m_2 \in \mathrm{BF}_{YZ}$. In that case

$$\begin{aligned}(m \circledcirc m_i)^{\downarrow Z} &= ((m \circledcirc m_i)^{\downarrow YZ})^{\downarrow Z} \quad \text{Lemma 2.5}\\ &= (m^{\downarrow YZ} \circledcirc m_i)^{\downarrow Z} \quad \text{Lemma 2.6}\end{aligned}$$

If furthermore $(m_1, m_2) \in R^Z(m)$, then:

$$(m^{\downarrow YZ} \circledcirc m_1)^{\downarrow Z} = (m^{\downarrow YZ} \circledcirc m_2)^{\downarrow Z} \tag{A.4}$$

Then using Lemma 2.6, we get:

$$\begin{aligned}(m\circledcap m_i)^{\downarrow XZ} &= (m^{\downarrow XZ}\circledcap m^{\downarrow YZ}\overline{\circledcap} m^{\downarrow Z}\circledcap m_i)^{\downarrow XZ} = m^{\downarrow XZ}\circledcap(m^{\downarrow YZ}\overline{\circledcap} m^{\downarrow Z}\circledcap m_i)^{\downarrow Z}\\ &= m^{\downarrow XZ}\overline{\circledcap} m^{\downarrow Z}\circledcap(m^{\downarrow YZ}\circledcap m_i)^{\downarrow Z}.\end{aligned}$$

Using Eq. (A.4), we get thus: $(m\circledcap m_1)^{\downarrow XZ} = (m\circledcap m_2)^{\downarrow XZ}$, hence $\mathrm{IR}_m(X,Y|Z)$.

We must now prove that $X \perp_m Y|Z$ implies that irrelevance preservation under $\circledcap$. By Lemma 3.1 of [22], Property 2, we have:

$$X \perp_m Y|Z \text{ iff } \exists f_1 \in \mathrm{BF}_{XZ}, f_2 \in \mathrm{BF}_{YZ} \text{ and } m\overline{\circledcap} m^{\downarrow Z} = f_1\circledcap f_2.$$

It can equally be written with $g \in \mathrm{BF}_{XZ}, h \in \mathrm{BF}_{YZ}, k \in \mathrm{BF}_Z$ as:

$$m\overline{\circledcap} m^{\downarrow Z} = g\circledcap h\overline{\circledcap} k$$

by $\circledcap$ combining any of f_1, f_2 with $m^{\downarrow Z}$ and k.

Suppose $m_1, m_2 \in \mathrm{BF}_{XYZ}$ and $X \perp_{m_1} Y|Z$, $X \perp_{m_2} Y|Z$. We must show we also have $X \perp_{m_1\circledcap m_2} Y|Z$. By $X \perp_{m_i} Y|Z$, we have:

$$m_i = m_i^{\downarrow XZ}\circledcap m_i^{\downarrow YZ}\overline{\circledcap} m_i^{\downarrow Z}$$

and

$$m_{12} = m_1\circledcap m_2 = (m_1^{\downarrow XZ}\circledcap m_2^{\downarrow XZ})\circledcap(m_1^{\downarrow YZ}\circledcap m_2^{\downarrow YZ})\overline{\circledcap}(m_1^{\downarrow Z}\circledcap m_2^{\downarrow Z})$$

hence $X \perp_{m_1\circledcap m_2} Y|Z$, which itself implies $\mathrm{IR}_{m_1\circledcap m_2}(X,Y|Z)$ as shown in the first part of this proof. $\square$

Proof of Theorem 8.1. The following properties prove the validity of the graphoid axioms for any belief function. In all proofs, we use the definition of conditional non-interactivity and we omit to indicate the domain XYZ. $\square$

Property P1. *Symmetry holds for any belief function distribution.*

Proof. Given three variables X, Y and Z.

We have to show $X \perp_m Y|Z \Longleftrightarrow Y \perp_m X|Z$.

$$\begin{aligned}&X \perp_m Y|Z \quad \text{(by Definition 5.1)}\\ &\Longleftrightarrow\ m\circledcap m^{\downarrow Z} = m^{\downarrow XZ}\circledcap m^{\downarrow YZ}\\ &\Longleftrightarrow\ m\circledcap m^{\downarrow Z} = m^{\downarrow YZ}\circledcap m^{\downarrow XZ} \quad \text{(commutativity of combination)}\\ &\Longleftrightarrow\ Y \perp_m X|Z. \qquad \square\end{aligned}$$

Property P2. *Decomposition holds for any belief function distribution.*

Proof. Given four variables X, Y, Z, and W.

We show that $X \perp_m Y \cup W|Z \Rightarrow X \perp_m Y|Z$.

$$\begin{aligned}
&X \perp_m Y \cup W|Z \quad \text{(by Definition 5.1)}\\
&\iff m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow (WY)Z}\\
&\iff m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow WYZ}.
\end{aligned}$$

We marginalize on W, we obtain:

$$\begin{aligned}
&(m \circledcirc m^{\downarrow Z})^{\downarrow W} = (m^{\downarrow XZ} \circledcirc m^{\downarrow WYZ})^{\downarrow W}\\
&\Rightarrow m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow YZ} \quad \text{(using CM1 : Lemma 2.6)}\\
&\Rightarrow X \perp_m Y|Z. \qquad \square
\end{aligned}$$

Property P3. *Weak union holds for any belief function distribution.*

Proof. Given four variables X, Y, Z and W.

$X \perp_m Y \cup W|Z \Rightarrow X \perp_m Y|W \cup Z$ is shown as follows:Using Definition 5.1, $X \perp_m Y \cup W|Z$

$$\begin{aligned}
&\iff m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow (WY)Z}\\
&\iff m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow WYZ}.
\end{aligned}$$

Combining $m^{\downarrow W}$ with both sides of the preceding equality, we get:

$$\begin{aligned}
&\iff (m \circledcirc m^{\downarrow Z}) \circledcirc m^{\downarrow W} = (m^{\downarrow XZ} \circledcirc m^{\downarrow WYZ}) \circledcirc m^{\downarrow W}\\
&\Rightarrow m \circledcirc (m^{\downarrow Z} \circledcirc m^{\downarrow W}) = (m^{\downarrow XZ} \circledcirc m^{\downarrow W}) \circledcirc m^{\downarrow WYZ}\\
&\Rightarrow m \circledcirc m^{\downarrow WZ} = m^{\downarrow WXZ} \circledcirc m^{\downarrow WYZ}\\
&\Rightarrow X \perp_m Y|W \cup Z. \qquad \square
\end{aligned}$$

Property P4. *Contraction holds for any belief function distribution.*

Proof. Given four variables X, Y, Z and W.

We have to show

$$X \perp_m Y|Z \text{ and } X \perp_m W|Y \cup Z \quad \Rightarrow \quad X \perp_m Y \cup W|Z$$

Using $X \perp_m Y|Z$ and $X \perp_m W|Y \cup Z$, we find

$$\begin{aligned}
&\Rightarrow m \circledcirc m^{\downarrow YZ} = m^{\downarrow XYZ} \circledcirc m^{\downarrow YWZ}\\
&\Rightarrow m \circledcirc (m^{\downarrow Z} \circledcirc m^{\downarrow Y}) = (m^{\downarrow XZ} \circledcirc m^{\downarrow Y}) \circledcirc m^{\downarrow YWZ}\\
&\Rightarrow (m \circledcirc m^{\downarrow Z}) \circledcirc m^{\downarrow Y} = (m^{\downarrow XZ} \circledcirc m^{\downarrow YWZ}) \circledcirc m^{\downarrow Y}\\
&\Rightarrow m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow YWZ}\\
&\Rightarrow X \perp_m Y \cup W|Z. \qquad \square
\end{aligned}$$

Property P5. *Intersection holds for any belief function distribution.*

Proof. Given four variables X, Y, Z and W.

We have to show

$$X \perp_m Y|Z \cup W \text{ and } X \perp_m W|Z \cup Y \quad \Rightarrow \quad X \perp_m Y \cup W|Z.$$

We have $X \perp_m Y|W \cup Z$

$$\begin{aligned} &\Rightarrow \quad m \circledcirc m^{\downarrow WZ} = m^{\downarrow WXZ} \circledcirc m^{\downarrow WYZ} \\ &\Rightarrow \quad (m \circledcirc m^{\downarrow WZ})^{\downarrow W} = (m^{\downarrow WXZ} \circledcirc m^{\downarrow WYZ})^{\downarrow W} \\ &\Rightarrow \quad m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow YZ}. \end{aligned} \tag{A.5}$$

We have also $X \perp_m W|Y \cup Z$

$$\begin{aligned} &\Rightarrow \quad m \circledcirc m^{\downarrow YZ} = m^{\downarrow YXZ} \circledcirc m^{\downarrow WYZ} \\ &\Rightarrow \quad (m \circledcirc m^{\downarrow YZ})^{\downarrow Y} = (m^{\downarrow YXZ} \circledcirc m^{\downarrow WYZ})^{\downarrow Y} \\ &\Rightarrow \quad m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow WZ}. \end{aligned} \tag{A.6}$$

Since we have the same left-hand sides of the preceding two equalities (A.5) and (A.6), the right-hands must be equal, so we obtain:

$$m^{\downarrow XZ} \circledcirc m^{\downarrow YZ} = m^{\downarrow XZ} \circledcirc m^{\downarrow WZ}.$$

When we combine $m^{\downarrow Y} \circledcirc m^{\downarrow W}$ with both sides of the preceding equality, we get:

$$\Rightarrow \quad m^{\downarrow XZ} \circledcirc m^{\downarrow YZ} \circledcirc m^{\downarrow Y} \circledcirc m^{\downarrow W} = m^{\downarrow XZ} \circledcirc m^{\downarrow WZ} \circledcirc m^{\downarrow Y} \circledcirc m^{\downarrow W}. \tag{A.7}$$

We marginalize (A.7) on Y, we obtain then:

$$\begin{aligned} &\Rightarrow \quad m^{\downarrow XZ} \circledcirc m^{\downarrow W} \circledcirc (m^{\downarrow YZ} \circledcirc m^{\downarrow Y}) = m^{\downarrow XZ} \circledcirc (m^{\downarrow WZ} \circledcirc m^{\downarrow W}) \circledcirc m^{\downarrow Y} \\ &\Rightarrow \quad (m^{\downarrow XZ} \circledcirc m^{\downarrow W}) \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc (m^{\downarrow WZ} \circledcirc m^{\downarrow W}) \\ &\Rightarrow \quad m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow WZ} \\ &\Rightarrow \quad X \perp_m W|Z. \end{aligned}$$

By Eq. (A.5), we have already proved that $X \perp_m Y|W \cup Z$

$$\begin{aligned} &\Rightarrow \quad m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow YZ} \\ &\Rightarrow \quad m \circledcirc m^{\downarrow Z} \circledcirc m^{\downarrow WYZ} = m^{\downarrow XZ} \circledcirc m^{\downarrow YZ} \circledcirc m^{\downarrow WYZ} \\ &\Rightarrow \quad m \circledcirc m^{\downarrow Z} \circledcirc (m^{\downarrow WY} \circledcirc m^{\downarrow Z}) = m^{\downarrow XZ} \circledcirc m^{\downarrow YZ} \circledcirc (m^{\downarrow WYZ} \circledcirc m^{\downarrow WY}) \\ &\Rightarrow \quad (m \circledcirc m^{\downarrow Z}) \circledcirc m^{\downarrow WY} = m^{\downarrow XZ} \circledcirc (m^{\downarrow YZ} \circledcirc m^{\downarrow WYZ}) \circledcirc m^{\downarrow WY} \\ &\Rightarrow \quad (m \circledcirc m^{\downarrow Z}) \circledcirc m^{\downarrow WY} = (m^{\downarrow XZ} \circledcirc m^{\downarrow WYZ}) \circledcirc m^{\downarrow WY} \\ &\Rightarrow \quad m \circledcirc m^{\downarrow Z} = m^{\downarrow XZ} \circledcirc m^{\downarrow WYZ} \\ &\Rightarrow \quad X \perp_m Y \cup W|Z. \qquad \square \end{aligned}$$

Proof of Theorem 8.2. Replace Z by X in the definition of non-interactivity, and use $XX = X$. One gets:

$$m^{XY} \textcircled{\cap} m^{XY\downarrow X} = m^{XY\downarrow X} \textcircled{\cap} m^{XY\downarrow XY}.$$

As $m^{XY\downarrow XY} = m^{XY}$, the theorem is proved. $\square$

Appendix B. Matricial representations and proofs

B.1. Relation between q and pl

The next relation holds between plausibility functions and commonality functions.

Lemma B.1

$$q(A) = -\sum_{B \subseteq A} (-1)^{|B|} \mathrm{pl}(B) \quad \text{for } A \neq \emptyset, \tag{B.1}$$

$$\mathrm{pl}(A) = -\sum_{B \subseteq A} (-1)^{|B|} q(B). \tag{B.2}$$

B.2. Solution by continuity for dogmatic belief functions

A *dogmatic belief function* is defined as a belief function with $m(\Omega) = 0$ [24]. Many theorems are easy to prove for non-dogmatic belief functions. A method to solve the dogmatic case consists in studying a solution when $m(\Omega) = \epsilon$, and then taking its limits when $\epsilon \to 0$. This is satisfactory if we accept that belief functions satisfy the continuity assumptions.

Axiom B.1 (*Continuity*). Let bel_ϵ be a family of belief functions indexed by the ϵ parameter. We assume that $\mathrm{bel}_0 = \lim_{\epsilon \to 0} \mathrm{bel}_\epsilon$.

This axiom simplifies the solution of rank problems in Section B.9. It could be avoided but proofs would be more complex.

B.3. Matricial notations

In order to simplify the proofs, we use the following matricial notations.

- For $m_i \in \mathrm{BF}_{YZ}$, $\mathbf{m}_i$ is the $2^{|YZ|}$ column vector with components $m_i(B), B \subseteq YZ$.
- For $m \in \mathrm{BF}_{XYZ}$, $\mathbf{P}$ is the $2^{|XZ|} \times 2^{|YZ|}$ matrix with components $p(A, B) = \mathrm{pl}(A^{\uparrow XYZ} \cap B^{\uparrow XYZ})$, $A \subseteq XZ, B \subseteq YZ$. As $\mathrm{pl}(\emptyset) = 0$, the components of $\mathbf{P}$ are 0 whenever $A^{\uparrow XYZ} \cap B^{\uparrow XYZ} = \emptyset$.

- For $m \in \mathrm{BF}_{XYZ}$, $\mathbf{H_X}$ is the $2^{|XZ|} \times 2^{|Z|}$ matrix with components $h_X(A,z) = \mathrm{pl}(A^{\uparrow XYZ} \cap z^{\uparrow XYZ})$, $A \subseteq XZ, z \in Z$. The columns of $\mathbf{H_X}$ are the columns of $\mathbf{P}$, where $B = z^{\uparrow XZ}$.
- For $m \in \mathrm{BF}_{XYZ}$, $\mathbf{H_Y}$ is the $2^{|YZ|} \times 2^{|Z|}$ matrix with components $h_Y(B,z) = \mathrm{pl}(B^{\uparrow XYZ} \cap z^{\uparrow XYZ}), B \subseteq YZ, z \in Z$. The columns of $\mathbf{H_Y}$ are the lines of $\mathbf{P}$, where $A = z^{\uparrow YZ}$.
- Let $\mathbf{N_X}$ be the $2^{|XZ|} \times 2^{|XZ|}$ matrix with components $n_X(A,B) = -(-1)^{|B|}$ if $B^{\uparrow XYZ} \subseteq A^{\uparrow XYZ}$, $= 0$ otherwise, where $A, B \subseteq XZ$.
- Let $\mathbf{N_Y}$ be the $2^{|YZ|} \times 2^{|YZ|}$ matrix with components $n_Y(A,B) = -(-1)^{|B|}$ if $B^{\uparrow XYZ} \subseteq A^{\uparrow XYZ}$, $= 0$ otherwise, where $A, B \subseteq YZ$.
- For $m \in \mathrm{BF}_{XYZ}$, $\mathbf{Q_X}$ is the $2^{|XZ|} \times 2^{|Z|}$ matrix with components $q_X(A,z) = q^{\downarrow XZ}[z](A)$, $A \subseteq XZ$, $z \in Z$. We have:

Lemma B.2. $\mathbf{Q_X} = \mathbf{N_X} \cdot \mathbf{H_X}$.

Proof. We have to prove that for $A \subseteq XZ$:

$$\begin{aligned} q^{\downarrow XZ}[z](A) &= \sum_{C \subseteq XZ} n_X(A,C) h_X(C,z) = \sum_{C \subseteq A} -(-1)^{|C|} \mathrm{pl}(C^{\uparrow XYZ} \cap z^{\uparrow XYZ}) \\ &= \sum_{C \subseteq A} -(-1)^{|C|} \mathrm{pl}^{\downarrow XZ}[z](C) \end{aligned}$$

what is true by Lemma B.1. □

- For $m \in \mathrm{BF}_{XYZ}$, $\mathbf{Q_Y}$ is the $2^{|YZ|} \times 2^{|Z|}$ matrix with components $q_Y(B,z) = q^{\downarrow YZ}[z](B)$, $B \subseteq YZ, z \in Z$. We have:

Lemma B.3. $\mathbf{Q_Y} = \mathbf{N_Y} \cdot \mathbf{H_Y}$.

Proof. The proof proceeds as for Lemma B.2. □

- Let $\mathbf{N}$ be the $2^{|XZ|} \times 2^{|YZ|}$ matrix with components $n(A,B) = -(-1)^{|B|}$ if $B \subseteq A$, $= 0$ otherwise, where $A \subseteq XZ, B \subseteq YZ$. It is inspired by the matrix that transforms a plausibility vector into a commonality vector (see Lemma B.1).

B.4. Indistinguishability

We rewrite Theorem 6.1 in matricial notation.

Theorem B.1. *Let* $m \in \mathrm{BF}_{XYZ}$, *and* $m_1, m_2 \in \mathrm{BF}_{YZ}$. *The following assertions are equivalent.*

$$\begin{aligned} &1.\ (m_1, m_2) \in R^Z(m). \\ &2.\ \mathbf{H'_Y} \cdot \mathbf{m_1} = \mathbf{H'_Y} \cdot \mathbf{m_2}. \end{aligned} \tag{B.3}$$

Proof. This theorem is just a rewriting of relation (14) of Theorem 6.1. □

Lemma B.4. $\mathrm{rank}(\mathbf{H_Y}) \leqslant 2^{|Z|} - 1$.

Proof. There as $2^{|Z|}$ possible values for z but the column of $\mathbf{H_Y}$ corresponding to $z = \emptyset$ is made of 0's, so the rank can never be larger than $2^{|Z|} - 1$. □

B.5. Irrelevance in matricial notation

Theorem B.2. *Let* $m \in \mathrm{BF}_{XYZ}$ *such that* $\mathrm{IR}_m(X, Y|Z)$. *Then for all* $m_1, m_2 \in \mathrm{BF}_{YZ}$ *with* $(m_1, m_2) \in R^Z(m)$, *we have*:

$$\mathbf{P} \cdot \mathbf{m_1} = \mathbf{P} \cdot \mathbf{m_2}$$

and $\mathbf{P} = \mathbf{C} \cdot \mathbf{H'_Y}$, *where* $\mathbf{C}$ *is a* $2^{|XZ|} \times 2^{|Z|}$ *matrix.*

Proof. By Theorem B.1, $(m_1, m_2) \in R^Z(m)$ iff $\mathbf{H'_Y} \cdot \mathbf{m_1} = \mathbf{H'_Y} \cdot \mathbf{m_2}$.
By Definition 6.2, $\mathrm{IR}_m(X, Y|Z)$ iff

$$(m \circledcirc m_1)^{\downarrow XZ} = (m \circledcirc m_2)^{\downarrow XZ}.$$

For $i = 1, 2$, $(m \circledcirc m_i)^{\downarrow XZ}$ can be written in term of its plausibility functions. For $A \subseteq XZ$,

$$\begin{aligned}(\mathrm{pl} \circledcirc \mathrm{pl}_i)^{\downarrow XZ}(A) &= (\mathrm{pl} \circledcirc \mathrm{pl}_i)(A^{\uparrow XYZ}) = \sum_{B \subseteq YZ} \mathrm{pl}[B^{\uparrow XYZ}](A^{\uparrow XYZ}) m_i(B) \\ &= \sum_{B \subseteq YZ} \mathrm{pl}(A^{\uparrow XYZ} \cap B^{\uparrow XYZ}) m_i(B) = (\mathbf{P} \cdot \mathbf{m_i})(A).\end{aligned}$$

So the equality $(m \circledcirc m_1)^{\downarrow XZ} = (m \circledcirc m_2)^{\downarrow XZ}$ is equivalent to $\mathbf{P} \cdot \mathbf{m_1} = \mathbf{P} \cdot \mathbf{m_2}$. Thus we have $\mathrm{IR}_m(X, Y|Z)$ iff

$$\forall m_1, m_2 \in \mathrm{BF}_{YZ} \text{ such that } \mathbf{H'_Y} \cdot \mathbf{m_1} = \mathbf{H'_Y} \cdot \mathbf{m_2}, \text{ we have } \mathbf{P} \cdot \mathbf{m_1} = \mathbf{P} \cdot \mathbf{m_2}.$$

This last constraint implies that $\mathbf{H'_Y}$ is a basis for $\mathbf{P}$, i.e., all lines of $\mathbf{P}$ are linear combinations of the lines of $\mathbf{H'_Y}$. So we can write $\mathbf{P} = \mathbf{C} \cdot \mathbf{H'_Y}$, where $\mathbf{C}$ is a $2^{|XZ|} \times 2^{|Z|}$ matrix. □

Theorem B.3 (Conditional irrelevance: alternate definition). $\mathrm{IR}_m(X, Y|Z)$ *with* $m \in \mathrm{BF}_{XYZ}$ *iff*

$$\forall m_1, m_2 \in \mathrm{BF}_{YZ}, \textit{ if } \mathbf{H'_Y} \cdot \mathbf{m_1} = \mathbf{H'_Y} \cdot \mathbf{m_2}, \textit{ then } \mathbf{P} \cdot \mathbf{m_1} = \mathbf{P} \cdot \mathbf{m_2}.$$

Proof. This property is just a rephrasing of the initial definition, using the properties derived in Theorem B.2. □

B.6. Symmetry of conditional irrelevance

In order to prove the symmetry of irrelevance, we first prove the following properties.

Theorem B.4 (Rank properties). *Let* $m \in \mathrm{BF}_{XYZ}$ *and* $\mathrm{IR}_m(X, Y|Z)$. *Then* $\mathrm{rank}(\mathbf{P}) = \mathrm{rank}(\mathbf{H_Y})$.

Proof. All lines of $\mathbf{P}$ being linear combinations of those in $\mathbf{H}'_{\mathbf{Y}}$, $\mathrm{rank}(\mathbf{P}) \leqslant \mathrm{rank}(\mathbf{H_Y})$. As all rows of $\mathbf{H}'_{\mathbf{Y}}$ are rows of $\mathbf{P}$, $\mathrm{rank}(\mathbf{H_Y}) \leqslant \mathrm{rank}(\mathbf{P})$. Therefore $\mathbf{P}$ and $\mathbf{H_Y}$ have the same rank. □

Theorem B.5 (Rank properties). *Let* $m \in \mathrm{BF}_{XYZ}$ *and* $\mathrm{IR}_m(X, Y|Z)$, *then* $\mathrm{rank}(\mathbf{P}) = \mathrm{rank}(\mathbf{H_X})$.

Proof. Let $r = \mathrm{rank}(\mathbf{P}) = \mathrm{rank}(\mathbf{H_Y})$.

Let

$$\mathbf{H}'_{\mathbf{Y}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

where

$$[\mathbf{A} \quad \mathbf{B}]$$

are r linearly independent rows of $\mathbf{H}'_{\mathbf{Y}}$, and

$$[\mathbf{C} \quad \mathbf{D}]$$

are $2^{|Z|} - r$ rows of $\mathbf{H}'_{\mathbf{Y}}$ that are linearly dependent of those in

$$[\mathbf{A} \quad \mathbf{B}].$$

Furthermore the columns of $\mathbf{A}$ are selected as those encountered in $\mathbf{H_X}$.

Then $\mathbf{P}$ admits a representation as

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{bmatrix}.$$

As $\mathrm{rank}(\mathbf{P}) = r$, all rows in

$$[\mathbf{E} \quad \mathbf{F}]$$

are linearly dependent of those in

$$[\mathbf{A} \quad \mathbf{B}].$$

Hence all rows of

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{E} \end{bmatrix}$$

are linearly dependent of those in

$$[\mathbf{A}],$$

and therefore

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \\ \mathbf{E} \end{bmatrix}\right) = r.$$

But this last matrix is just $\mathbf{H_X}$, hence $\operatorname{rank}(\mathbf{H_X}) = r$. □

Proof of Theorem 6.2. $\mathrm{IR}_m(X, Y|Z)$ iff $\mathrm{IR}_m(Y, X|Z)$.

Proof. Let $\mathbf{P}$ be represented as

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

where

$$[\mathbf{A} \quad \mathbf{B}] = \mathbf{H}'_\mathbf{Y},$$

and

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = \mathbf{H_X}.$$

As $\operatorname{rank}(\mathbf{H_X}) = \operatorname{rank}(\mathbf{H_Y})$, we know that all columns of

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}$$

can be represented as linear combination of those in

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix},$$

therefore once $\mathbf{m}'_1\mathbf{H_X} = \mathbf{m}'_2\mathbf{H_X}$, we also have

$$\mathbf{m}'_1\begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} = \mathbf{m}'_2\begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}.$$

Hence we have $\mathbf{m}'_1\mathbf{P} = \mathbf{m}'_2\mathbf{P}$. □

B.7. Links with marginal irrelevance

We prove Theorem 6.3 that shows that conditional irrelevance degrades into marginal irrelevance when $|Z| = 1$.

Proof of Theorem 6.3. If $|Z| = 1$, $\mathrm{IR}_m(X, Y|Z) = \mathrm{IR}_m(X, Y)$.

Proof. Let $A \subseteq XZ, B \subseteq YZ$. The constraint $\mathbf{P} = \mathbf{C} \cdot \mathbf{H}'_{\mathbf{Y}}$ can be written as:

$$\mathrm{pl}(A^{\uparrow XYZ} \cap B^{\uparrow XYZ}) = \sum_{z \subseteq Z} c(A, z)\mathrm{pl}(z^{\uparrow XYZ} \cap B^{\uparrow XYZ}) = \sum_{z \subseteq Z} c(A, z)\mathrm{pl}[z]^{\downarrow YZ}(B).$$

When $|Z| = 1$, A can be written as x, B as y, and the constraints become:

$$\mathrm{pl}(x^{\uparrow XY} \cap y^{\uparrow XY}) = \mathrm{pl}[y]^{\downarrow X}(x) = c(y)\mathrm{pl}^{\downarrow X}(x) \quad \forall x \subseteq X, y \subseteq Y.$$

Hence $\mathrm{pl}[y]^{\downarrow X} \propto \mathrm{pl}^{\downarrow X}$, and $\mathrm{IR}_m(X, Y)$ holds. $\square$

B.8. Conditioning on Z and irrelevance

We prove the technical Theorem 6.4, about a particular belief function.

Proof of Theorem 6.4. If $m(z_0^{\uparrow XYZ}) = 1$ for $z_0 \subseteq Z$, then $\mathrm{IR}_m(X, Y|Z)$.

Proof. We keep the notation convention that $A \subseteq XZ$, $B \subseteq YZ$ and $z \subseteq Z$. Let $\mathbf{H} = [h(z, B)]$ where $h(z, B) = \mathrm{pl}(z^{\uparrow XYZ} \cap B^{\uparrow XYZ}) = 1$ if $z^{\uparrow XYZ} \cap B^{\uparrow XYZ} \cap z_0^{\uparrow XYZ} \neq \emptyset$, and $= 0$ otherwise.

Let $m_1, m_2 \in \mathrm{BF}_{YZ}$ with $\mathbf{H} \cdot \mathbf{m_1} = \mathbf{H} \cdot \mathbf{m_2}$. We have:

$$\begin{aligned}(\mathbf{H} \cdot \mathbf{m_i})(z) &= \sum_{B \subseteq YZ} h(z, B)m_i(B) = \sum_{B: z^{\uparrow XYZ} \cap B^{\uparrow XYZ} \cap z_0^{\uparrow XYZ} \neq \emptyset} m_i(B) \\ &= \sum_{B: z^{\uparrow YZ} \cap B \cap z_0^{\uparrow YZ} \neq \emptyset} m_i(B) = \mathrm{pl}_i((z \cap z_0)^{\uparrow YZ}).\end{aligned}$$

So $\mathbf{H} \cdot \mathbf{m_1} = \mathbf{H} \cdot \mathbf{m_2}$ implies

$$\mathrm{pl}_1(z^{\uparrow YZ}) = \mathrm{pl}_2(z^{\uparrow YZ}) \quad \forall z \subseteq z_0. \tag{B.4}$$

We proceed with the $\mathbf{P} = [p(A, B)]$ matrix, where $p(A, B) = \mathrm{pl}(A^{\uparrow XYZ} \cap B^{\uparrow XYZ}) = 1$ if $A^{\uparrow XYZ} \cap B^{\uparrow XYZ} \cap z^{\uparrow XYZ} \neq \emptyset$, and $= 0$ otherwise. We have:

$$\begin{aligned}(\mathbf{p} \cdot \mathbf{m_i})(A) &= \sum_{B \subseteq YZ} p(A, B)m_i(B) = \sum_{B: A^{\uparrow XYZ} \cap B^{\uparrow XYZ} \cap z^{\uparrow XYZ} \neq \emptyset} m_i(B) \\ &= \sum_{B: B^{\downarrow Z} \cap (A^{\downarrow Z} \cap z_0) \neq \emptyset} m_i(B) = \mathrm{pl}_i((A^{\downarrow Z} \cap z_0)^{\uparrow YZ}).\end{aligned}$$

From (B.4) we have $\mathrm{pl}_1((A^{\downarrow Z} \cap z_0)^{\uparrow YZ}) = \mathrm{pl}_2((A^{\downarrow Z} \cap z_0)^{\uparrow YZ})$ for all $A \subseteq YZ$, as all that counts is the projection of A on Z. Hence the requirement $\mathbf{P} \cdot \mathbf{m_1} = \mathbf{P} \cdot \mathbf{m_2}$ is satisfied and we have $\mathrm{IR}_m(X, Y|Z)$. $\square$

B.9. The commonality matrix R

By pre- and post-multiplying $\mathbf{P}$ by $\mathbf{N_X}$ and $\mathbf{N'_Y}$, the result is a matrix made of 'rectangles' located on the diagonal.

Theorem B.6. *Let $m \in \mathrm{BF}_{XYZ}$ and $\mathrm{IR}_m(X, Y|Z)$. $\mathbf{R} = \mathbf{N_X} \cdot \mathbf{P} \cdot \mathbf{N'_Y}$ is a matrix which components satisfy*:

1. $r(A,B) = (-1)^{1+|z^*|} q(A^{\uparrow XYZ} \cap B^{\uparrow XYZ})$, *if* $A^{\downarrow Z} = B^{\downarrow Z} = z^* \neq \emptyset$,
2. $r(A,B) = 0$, whenever $A^{\downarrow Z} \neq B^{\downarrow Z}$,
3. $r(\emptyset, \emptyset) = 0$.

Proof. We keep the notation convention that $A \subseteq XZ$, $B \subseteq YZ$ and $z \subseteq Z$.

1. (a) Let $A^{\downarrow Z} = B^{\downarrow Z} = z^*$. We have $A = \bigcup_{z_i \in z^*} (x_{i,A}, z_i)$, $B = \bigcup_{z_i \in z^*} (y_{i,B}, z_i)$, and $A^{\uparrow XYZ} \cap B^{\uparrow XYZ} = \bigcup_{z_i \in z^*} (x_{i,A}, y_{i,B}, z_i)$, where $x_{i,A} \subseteq X$, $y_{i,B} \subseteq Y$. Then:

$$r(A,B) = \sum_{C \subseteq XZ} \sum_{D \subseteq YZ} n_X(A,C) p(C,D) n_Y(B,D) = \sum_{C \subseteq A} \sum_{D \subseteq B} (-1)^{|C|+|D|} p(C,D)$$
$$= \sum_{\emptyset \neq C \subseteq A, \emptyset \neq D \subseteq B, C^{\downarrow Z} \cap D^{\downarrow Z} \neq \emptyset} (-1)^{|C|+|D|} \mathrm{pl}(C^{\uparrow XYZ} \cap D^{\uparrow XYZ}).$$

(b) For $w \subseteq (X, Y, z^*)$, we have:

$$m[z^*](w) = \sum_{w' \subseteq \overline{(X,Y,z^*)}} m(w \cup w').$$

We know that m's focal elements belong to ZLR (as m satisfies $\mathrm{IR}_m(X, Y|Z)$), hence they can be represented as $\bigcup_{z_i \in Z} (x_i, y_i, z_i)$.

The only non-zero terms for $m[z^*](w)$ are those given to a $w \subseteq (X, Y, z^*)$ that admits a representation as: $w = \bigcup_{z_i \in (X,Y,z^*)} (x_{i,w}, y_{i,w}, z_i)$ where $x_{i,w} \subseteq X$, $y_{i,w} \subseteq Y$.

We successively consider how many times (they turn out to be +1, −1 or 0) the terms $m[z^*](\bigcup_{z_i \in z*} (x_{i,w}, y_{i,w}, z_i))$ are encountered when computing $r(A,B)$.

(c) Consider first the case $w^{\downarrow XZ} = A$, $w^{\downarrow YZ} = B$. This term is encountered in every $p(C,D)$ included in $r(A,B)$, provided $C^{\downarrow Z} \cap D^{\downarrow Z} \neq \emptyset$. Its coefficients are thus:

$$\sum_{\emptyset\neq C\subseteq A}(-1)^{|C|}\sum_{\emptyset\neq D\subseteq B, C^{\downarrow Z}\cap D^{\downarrow Z}\neq\emptyset}(-1)^{|D|}. \tag{B.5}$$

Given C, and with $B_1 = B\cap C^{\downarrow Z\uparrow YZ}, B_2 = B\cap\overline{B_1}$, the second summation is

$$\sum_{\emptyset\neq D_1\subseteq B_1}(-1)^{|D_1|}\sum_{D_2\subseteq B_2}(-1)^{|D_2|} = \begin{cases} 0 & \text{if } B_2\neq\emptyset,\\ \sum_{\emptyset\neq D_1\subseteq B_1}(-1)^{|D_1|} = -1 & \text{if } B_2=\emptyset.\end{cases}$$

The requirement $B_2=\emptyset$ is equivalent to $C^{\downarrow Z}=z^*$, that means also $B_1^{\downarrow Z}=z^*$.

Let $A=\bigcup_{z_i\in z^*}(x_{i,A},z_i)$, with every $x_{i,A}\neq\emptyset$. So (B.5) becomes:

$$\begin{aligned}\sum_{\emptyset\neq C\subseteq A, C^{\downarrow Z}=z^*}(-1)^{1+|C|} &= \sum_{\{x_i:\emptyset\neq x_i\subseteq x_{i,A}\}}(-1)^{1+\sum_i |x_i|}\\ &= -\sum_{\emptyset\neq x_1\subseteq x_{1,A}}(-1)^{|x_1|}\sum_{\emptyset\neq x_2\subseteq x_{2,A}}(-1)^{|x_2|}\ldots\sum_{\emptyset\neq x_{|z^*|}\subseteq x_{n,A}}(-1)^{|x_{|z^*|}|}\\ &= (-1)^{1+|z^*|}.\end{aligned}$$

(d) Consider now the mass $m[z^*](E^{XZ\uparrow XYZ}\cap F^{YZ\uparrow XYZ})$ with $E\subseteq A$, $F\subseteq B$, and at least one of $E\neq A$ or $F\neq B$ holds.

We have $E^{XZ\uparrow XYZ}\cap F^{YZ\uparrow XYZ}=\bigcup_{z_i\in z^*}(x_{i,E},y_{i,F},z_i)$, where $x_{i,E}\subseteq x_{i,A}$ and $y_{i,F}\subseteq y_{i,B}$.

We consider all terms $p(C,D)$, where $C^{XZ\uparrow XYZ}\cap D^{YZ\uparrow XYZ}$ has a non-empty intersection with $E^{XZ\uparrow XYZ}\cap F^{YZ\uparrow XYZ}$, and among them those that share the same intersection with $E^{XZ\uparrow XYZ}\cap F^{YZ\uparrow XYZ}$.

They admit a representation $\bigcup_{z_i\in z^*}((x_{i,C}\cap x_{i,E})\cup(x_{i,C}\cap\overline{x_{i,E}}),(y_{i,D}\cap y_{i,F})\cup(y_{i,D}\cap\overline{y_{i,F}}),z_i)$.

We fix $x_{i,C}\cap x_{i,E}$ and $y_{i,D}\cap y_{i,F}$, and we consider the sets obtained while letting $(x_{i,C}\cap\overline{x_{i,E}})$ varies with $\emptyset\subseteq x_{i,C}\cap\overline{x_{i,E}}\subseteq x_{i,A}\cap\overline{x_{i,E}}$.

Its coefficient is $(-1)^{\sum |x|}$. Either $x_{i,A}\cap\overline{x_{i,E}}=\emptyset$ and the coefficient is ± 1, or $x_{i,A}\cap\overline{x_{i,E}}\neq\emptyset$ and the coefficient is 0.

The only way to get a non-zero coefficient is achieved when all $x_{i,A}\cap\overline{x_{i,E}}=\emptyset$ and $y_{i,B}\cap\overline{y_{i,F}}=\emptyset$, i.e., when $A^{\uparrow XYZ}\cap B^{\uparrow XYZ}=E^{XZ\uparrow XYZ}\cap F^{YZ\uparrow XYZ}$ but this is back to the previous analysis and we have shown that the coefficient is then $(-1)^{|z^*|+1}$.

So $r(A,B)$ contains only $(-1)^{|z^*|}m[z^*](A^{\uparrow XYZ}\cap B^{\uparrow XYZ})$ which is equal to $(-1)^{|z^*|+1}q(A^{\uparrow XYZ}\cap B^{\uparrow XYZ})$ with $z^*=A^{\downarrow Z}=B^{\downarrow Z}$.

2. Let

- $z^*=A^{\downarrow Z}\cap B^{\downarrow Z}$,
- $A^*=A\cap z^{*\uparrow XZ}$, $A'=A\cap\overline{A^*}$, so $A=A^*\cup A'$.
- $B^*=B\cap z^{*\uparrow YZ}$, $B'=B\cap\overline{B^*}$, so $B=B^*\cup B'$.

We show that $r(A,B)=0$ whenever $A^{\downarrow Z}\neq B^{\downarrow Z}$. We had:

$$r(A,B) = \sum_{C\subseteq A}\sum_{D\subseteq B}(-1)^{|C|+|D|}p(C,D)$$

$$= \sum_{C_1\subseteq A^*}\sum_{C_2\subseteq A'}\sum_{D_1\subseteq B^*}\sum_{D_2\subseteq B'}(-1)^{|C|+|D|}p(C,D)$$

$$= \sum_{C_1\subseteq A^*}\sum_{D_1\subseteq B^*}p(C_1,D_1)(-1)^{|C_1|+|D_1|}\sum_{C_2\subseteq A'}\sum_{D_2\subseteq B'}(-1)^{|C_2|+|D_2|}$$

$$= \sum_{C_1\subseteq A^*}\sum_{D_1\subseteq B^*}p(C_1,D_1)(-1)^{|C_1|+|D_1|} \quad \text{if } A' = B' = \emptyset = 0 \text{ otherwise}$$

3. The case $r(\emptyset,\emptyset) = 0$ is direct as $r(\emptyset,\emptyset) = p(\emptyset,\emptyset) = 0$. □

Suppose the subsets A of XZ and B of YZ are identically ordered in the matrix $\mathbf{R}$ according to the values of their projection on Z. It means that we start with $\emptyset$, and put successively the subsets A of XZ which projection on Z is z_1, is z_2, is $\{z_1, z_2\}$, is $z_3, \ldots$ is Z, and similarly with the subsets of YZ. Theorem B.6 shows that R is then made of $2^{|Z|}-1$ non-zero blocks located along its 'diagonal'. As $\mathbf{N_X}$ and $\mathbf{N_Y}$ are non-singular, $\mathbf{P}$ and $\mathbf{R}$ have equal ranks. Therefore the basis of $\mathbf{R}$ is built by selecting in each block one line and one column. The choice is arbitrary, but the best choice consists in using the lines that correspond to those subsets of XZ which are equal to $z^{\uparrow XZ}$ for $z \subseteq Z$ (and the same for the columns). The resulting matrix are denoted $\mathbf{Q_X}$ and $\mathbf{Q_Y}$, respectively, as shown hereafter (see Lemmas B.2 and B.3):

$$\mathbf{N_X}\cdot\mathbf{H_X} = \mathbf{Q_X} \quad \text{and} \quad \mathbf{N_Y}\cdot\mathbf{H_Y} = \mathbf{Q_Y}.$$

Theorem B.7. *Let*
- $\mathbf{D_X} = [d_X(A,z)]$ *is a* $2^{|XZ|}\times 2^{|Z|}$ *matrix with* $d_X(A,z) = q^{\downarrow XZ}[z](A)$ if $A^{\downarrow Z} = z$, *and* $=0$ *otherwise,*
- $\mathbf{D_Y} = [d_Y(A,z)]$ *is a* $2^{|YZ|}\times 2^{|Z|}$ *matrix with* $d_Y(B,z) = q^{\downarrow YZ}[z](B)$ if $B^{\downarrow Z} = z$, *and* $=0$ *otherwise,*
- $\mathbf{F} = [f(z_1,z_2)]$ is a $2^{|Z|}\times 2^{|Z|}$ *diagonal matrix with* $f(z,z) = (-1)^{1+|z|}q^{\downarrow Z}(z)$.

Then $\mathbf{R} = \mathbf{D_X}\cdot\mathbf{F}\cdot\mathbf{D'_Y}$.

Proof. 1. Let $\mathbf{Q} = [q(z_1,z_2)]$ be the $2^{|Z|}\times 2^{|Z|}$ matrix with $q(z_1,z_2) = 1$ if $z_1 \subseteq z_2$, and $=0$ otherwise. Note that $\mathbf{Q}$ is the matrix that transforms a bba into a credibility function. So $\mathbf{Q^{-1}}$ is well defined, and its component $\overline{q}(z_1,z_2) = (-1)^{|z_2|-|z_1|}$ if $z_1 \subseteq z_2$, and $=0$ otherwise. By construction, $\mathbf{Q_X} = \mathbf{D_X}\mathbf{Q}$ and $\mathbf{Q_Y} = \mathbf{D_Y}\mathbf{Q}$.

2. We prove first that there exists a diagonal matrix $\mathbf{F}$ such that:

$$\mathbf{P} = \mathbf{H_X}\mathbf{Q}^{-1}\mathbf{F}\mathbf{Q}'^{-1}\mathbf{H}'_\mathbf{Y}. \tag{B.6}$$

Based on the theory of generalized inverse, we know that the $2^{|XZ|} \times 2^{|YZ|}$ matrix $\mathbf{P}$ of rank r can be represented as $\mathbf{P} = \mathbf{P_X}\mathbf{L}\mathbf{P}'_\mathbf{Y}$, where

- $\mathbf{P_X}$ is a $2^{|XZ|} \times r$ matrix of rank r,
- $\mathbf{P_Y}$ is a $2^{|YZ|} \times r$ matrix of rank r,
- $\mathbf{L}$ is a $r \times r$ diagonal matrix of rank r,
- $\mathbf{P}'_\mathbf{Y}\mathbf{P_X} = \mathbf{I}$, the $r \times r$ identity matrix.

We know also that $\text{rank}(\mathbf{H_X}) = \text{rank}(\mathbf{H_Y}) = r$. So there exists a $\mathbf{T_X}$ and a $\mathbf{T_Y}$ so that $\mathbf{P_X} = \mathbf{H_X}\mathbf{T_X}$ and $\mathbf{P_Y} = \mathbf{H_Y}\mathbf{T_Y}$. So $\mathbf{P}$ can also be represented as

$$\mathbf{P} = \mathbf{H_X}\mathbf{D}\mathbf{H}'_\mathbf{Y},$$

where $\mathbf{D} = \mathbf{T_X}\mathbf{L}\mathbf{T}'_\mathbf{Y}$ is a $2^{|Z|} \times 2^{|Z|}$ matrix.

Now we have

$$\begin{aligned}
\mathbf{R} &= \mathbf{N_X}\mathbf{P}\mathbf{N}'_\mathbf{Y} \\
&= \mathbf{N_X}\mathbf{H_X}\mathbf{D}\mathbf{H}'_\mathbf{Y}\mathbf{N}'_\mathbf{Y} && \text{(B.7)} \\
&= \mathbf{Q_X}\mathbf{D}\mathbf{Q}'_\mathbf{Y} \\
&= \mathbf{D_X}\mathbf{Q}\mathbf{D}\mathbf{Q}'\mathbf{D}'_\mathbf{Y} \\
&= \mathbf{D_X}\mathbf{F}\mathbf{D}'_\mathbf{Y}, && \text{(B.8)}
\end{aligned}$$

where $\mathbf{D_X}$ and $\mathbf{D_Y}$ are defined in the theorem and $\mathbf{F} = \mathbf{Q}\mathbf{D}\mathbf{Q}' = [f(z_1, z_2)]$, $z_1, z_2 \subseteq Z$, which values are still to be derived.

These equalities result from the general equality $q[D](C) = q(C)$ if $C \subseteq D$, and 0 otherwise.

3. Let $\mathbf{R} = [r(A,B)]$ with $A \subseteq XZ$ and $B \subseteq YZ$. We explore the value of $r(A,B)$. From (B.8), we have:

$$\begin{aligned}
r(A,B) &= \sum_{z_1,z_2 Z} d_X(A,z_1)f(z_1,z_2)d_Y(B,z_2) \\
&= q^{\downarrow XZ}[A^{\downarrow Z}](A)f(A^{\downarrow Z},B^{\downarrow Z})q^{\downarrow YZ}[B^{\downarrow Z}](B).
\end{aligned}$$

Suppose $A^{\downarrow Z} = B^{\downarrow Z} = z$, then, by Theorem B.6:

$$r(A,B) = q^{\downarrow XZ}[z](A)f(z,z)q^{\downarrow YZ}[z](B) = q^{\downarrow XZ}(A)f(z,z)q^{\downarrow YZ}(B).$$

If $A^{\downarrow Z} = z_A$ and $B^{\downarrow Z} = z_B$ with $z_A \neq z_B$, then $r(A,B) = 0$ by Theorem B.6. So:

$$0 = q^{\downarrow XZ}[z_A](A)f(z_A,z_B)q^{\downarrow YZ}[z_B](B).$$

Suppose that q is not dogmatic (i.e., $q(X,Y,Z) > 0$). Then the two q terms are positive, and thus $f(z_A,z_B) = 0$. If q is dogmatic, we just change q so that $q(X,Y,Z) = \epsilon$, and we find then that $f(z_A,z_B) = 0$. This is true for every ϵ, and by the continuity assumed in Axiom B.1, we can deduce that even for the dogmatic case, $f(z_A, z_B) = 0$ is required.

Thus $f(z_1, z_2) = 0$ whenever $z_1 \neq z_2$, what means that $\mathbf{F}$ is a diagonal matrix. Therefore replacing $\mathbf{D}$ by $\mathbf{Q}^{-1}\mathbf{F}\mathbf{Q}'^{-1}$ in (B.7) we get

$$\mathbf{P} = \mathbf{H_X}\mathbf{Q}^{-1}\mathbf{F}\mathbf{Q}'^{-1}\mathbf{H}'_{\mathbf{Y}},$$

where $\mathbf{F}$ is diagonal. Hence (B.6) is proved.

4. We prove that the diagonal elements of $\mathbf{F}$ are $(-1)^{|z|+1}/q^{\downarrow Z}(z)$ when the denominator is non-null, and $=0$ otherwise.

The rows of $\mathbf{H_X}$ can be permuted so that the rows which index is a cylindrical extension of a subset of Z are at the top. Then

$$\mathbf{H_X} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B_X} \end{bmatrix},$$

where the $(z_1^{\uparrow XZ}, z_2)$ element of $\mathbf{A}$ is $\mathrm{pl}(z_1^{\uparrow XYZ} \cap z_2^{\uparrow XYZ}) = \mathrm{pl}((z_1 \cap z_2)^{\uparrow XYZ})$. So $\mathbf{A}$ is symmetrical. The same operation can be done on $\mathbf{H_Y}$, and the upper block is in fact the same A as in the decomposition of $\mathbf{H_X}$. Applying the same permutation on $\mathbf{P}$, we find that its upper left corner is also the $\mathbf{A}$ matrix.

So we get

$$\mathbf{A} = \mathbf{A}\mathbf{Q}^{-1}\mathbf{F}\mathbf{Q}'^{-1}\mathbf{A}. \tag{B.9}$$

Let $\mathbf{B} = [b(z_1, z_2)]$ be the $2^{|Z|} \times 2^{|Z|}$ diagonal matrix with $b(z_1, z_2) = (-1)^{|z|+1}q^{\downarrow Z}(z)$ if $z_1 = z_2 = z \neq \emptyset$, and $=0$ otherwise. Then the element $\beta(z_1, z_2)$, $z_1, z_2 \subseteq Z$, of $\mathbf{Q}'\mathbf{B}\mathbf{Q}$ is:

$$\begin{aligned} \beta(z_1, z_2) &= \sum_{z_3 \subseteq Z, z_4 \subseteq Z} q(z_3, z_1) b(z_3, z_4) q(z_4, z_2) = \sum_{z_3 \subseteq z_1, z_4 \subseteq z_2} b(z_3, z_4) \\ &= \sum_{\emptyset \neq z \subseteq z_1 \cap z_2} b(z, z) = \sum_{\emptyset \neq z \subseteq z_1 \cap z_2} (-1)^{|z|+1} q^{\downarrow Z}(z) = \mathrm{pl}^{\downarrow Z}(z_1 \cap z_2) \\ &= \mathrm{pl}((z_1 \cap z_2)^{\uparrow XYZ}). \end{aligned}$$

Hence $\mathbf{Q}'\mathbf{B}\mathbf{Q} = \mathbf{A}$. Eq. (B.9) becomes $\mathbf{B} = \mathbf{BFB}$, where $\mathbf{B}$ and $\mathbf{F}$ are diagonal. So the diagonal elements of $\mathbf{F}$ are $(-1)^{|z|+1}/q^{\downarrow Z}(z)$ when denominator is non-null, and $=0$ otherwise (anything would be satisfactory, and we choose for simplicity sake). □

The next theorem shows that the second requirement requested by Theorem 5.4 is satisfied.

Theorem B.8. *For $A \subseteq XZ, B \subseteq YZ$, $X \perp\!\!\!\perp_m Y|Z$ implies*

$$q(A^{\uparrow XUZ} \cap B^{\uparrow YUZ}) = \frac{q^{XYZ\downarrow XZ}(w^{\downarrow XZ})q^{XYZ\downarrow YZ}(w^{\downarrow YZ})}{q^{XYZ\downarrow Z}(w^{\downarrow Z})}.$$

Proof. By Theorem B.6, the non-zero r terms in $\mathbf{R}$ are

$$r(A,B) = (-1)^{|z^*|+1} q(A^{\uparrow XUZ} \cap B^{\uparrow YUZ}) \quad \text{if } A^{\downarrow Z} = B^{\downarrow Z} = z^* \neq \emptyset.$$

By Theorem B.7, we found also

$$r(A,B) = (-1)^{|z^*|+1} \frac{q^{XYZ\downarrow XZ}(w^{\downarrow XZ}) q^{XYZ\downarrow YZ}(w^{\downarrow YZ})}{q^{XYZ\downarrow Z}(w^{\downarrow Z})}.$$

Hence the theorem is proved. □

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